Integer Linear Programming

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Outline

Standard form

Mixed integer linear programming

a general mixed integer linear programming (MILP) has two general forms

$$\begin{array}{ll} \text{minimize} & c^T x + d^T y \\ \text{subject to} & Ax + By = b \\ & x, y \succeq 0, \\ & x \text{ is integer} \end{array} \qquad \begin{array}{ll} \text{minimize} & c^T x + d^T y \\ \text{subject to} & Gx + Fy \preceq h \\ & Ax + By = b \\ & x \text{ is integer} \end{array}$$

- $x = (x_1, \ldots, x_n)$ where $x_i \in \mathbf{Z}$ (integer variables)
- $y = (y_1, \ldots, y_p) \in \mathbf{R}^p$ (real-valued variables)
- the objective function: $c^T x + d^T y = \sum_{i=1}^n c_i x_i + \sum_{i=1}^p d_i x_i$
- the objective function and constraint functions are *linear* in x and y

Modeling techniques

Common patterns

- binary choice (zero-one knapsack)
- forcing constraints (facility location problem)
- relation between variables
- disjunctive constraints
- restricted range of values
- arbitrary piecewise linear functions (set packing, set covering)
- sequencing problem with setup times

Binary choice

scenario: when to encode a choice between two alternativers, use binary variables

zero-one knapsack problem

$$\text{maximize} \;\; c^T x \;\; \text{subject to} \;\; w^T x \leq K, \;\; x_j \in \{0,1\}, \;\; j = 1,2,\ldots,n$$

setting:

- given n items; each has weight w_j and value c_j
- given a bound K on the total weight that can be carried in a knapsack

goal: decide to select items to maximize the total value

Example: Knapsack problem

description:

- given a backpack that can only hold up to 15 kg
- each item has a specific weight and a specific importance value
- decide which items to take to maximize total value without breaking the bag

1. **variables:** binary decision of each item (1 if taken, 0 if not taken); you cannot take 'half' of the camera
2. **objective:** maximize sum of total **value** of items you pack
3. **constraints:** the sum of the weights of selected items must be ≤ 15 kg
4. **problem parameters:** backpack weight limit, value of each item

NASA Example: bring equipments to a satellite

description:

- given a backpack that can only hold up to 50 kg
- each item has a specific weight and a specific importance value
- decide which items to take to maximize total value without breaking the weight limit

Forcing constraints

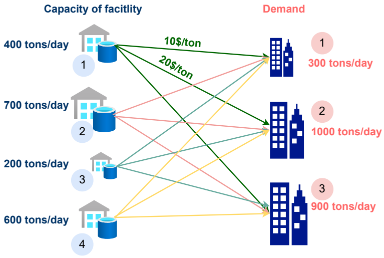
scenario: some variables are dependent; decision A can be made only if decision B has been made

modeling:

- introduce binary variables x, y corresponding to decision A and B, respectively
- set x to equal to 1 if A is chosen and 0 otherwise
- the dependence of the two decisions can be modeled using the constraint: $x \leq y$
- if $y = 0$ (decision B is not made), then $x = 0$ (decision A is not made)

(Capacitated) Facility location problem

given n facility locations indexed by i and m clients to be serviced indexed by j



- each client j has a demand d_j to be filled by selected facilities
- u_i is the capacity of facility i
- f_i is a fixed cost of opening facility i, while there is a cost c_{ij} of serving client j from facility i (\$/demand unit)

1. **goal:** select a set of facility locations and assign each client to facilities that minimizes the total cost while meeting the targeted demand
2. **variables:** $x_i \in \{0,1\}$ for $i = 1, 2, \ldots, m$ where $x_i = 1$ if facility i is open ; $y_{ij} \in \mathbf{R}$ represents the **fraction** of the demand d_j filled by facility i

(Capacitated) Facility location problem: formulation

$$
\begin{array}{lll}
\text{minimize} & \sum_{i=1}^n f_j x_i + \sum_{i=1}^n \sum_{j=1}^m c_{ij} d_j y_{ij} & \\
\text{subject to} & y_{ij} \geq 0, \quad i = 1, \ldots, n, j = 1, \ldots, m & \text{fractions are non-negative} \\
 & x_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n & \text{facility } i \text{ is either open or closed} \\
 & \sum_{i=1}^n y_{ij} = 1, \quad j = 1, 2, \ldots, m & \text{fractions of facilities to serve } j \text{ sum to } 1 \\
 & \sum_{j=1}^m d_j y_{ij} \leq u_i x_i, \quad i = 1, 2, \ldots, n & \text{max capacity of facility } i \\
 & & \text{force } y_{ij} \text{ to be positive when } x_i = 1
\end{array}
$$

in matrix form: $C, Y \in \mathbf{R}^{n \times m}$ and $x, d, u \in \mathbf{R}^n$

- cost: $f^T x + \mathbf{1}^T (C \odot Y) d$
- constraints:

$$
Y \geq 0, \quad x \in \{0, 1\}^n, \quad \mathbf{1}_n^T Y = \mathbf{1}_m, \quad Yd \leq u \odot x
$$

(Uncapacitated) Facility location problem

when all facilities has ∞ capacity,

- capacitated: optimization solely selects the facility with the lowest cost
- change: modify $y_{ij} \in \{0,1\}$ where $y_{ij} = 1$ if customer j is served by facility i

$$\begin{array}{lll}
\text{minimize} & \sum_{i=1}^n f_j x_i + \sum_{i=1}^n \sum_{j=1}^m c_{ij} d_j y_{ij} & \\
\text{subject to} & y_{ij} \in \{0,1\}, \quad i = 1,\ldots,n, j = 1,\ldots,m & \text{binary value} \\
& x_i \in \{0,1\}, \quad i = 1,2,\ldots,n & \text{facility } i \text{ is either open or closed} \\
& \sum_{i=1}^n y_{ij} = 1, \quad j = 1,2,\ldots,m & \text{to serve } j \text{ only pick one facility } i \\
& y_{ij} \leq x_i, \quad i = 1,2,\ldots,n, \; j = 1,2,\ldots,m & \text{use facility } i \text{ only if it's open} \\
& & \text{force } y_{ij} \text{ to be positive when } x_i = 1
\end{array}$$

in matrix form: $C, Y \in \mathbf{R}^{n \times m}$ and $x, d \in \mathbf{R}^n$

- cost: $f^T x + \mathbf{1}^T (C \odot Y) d$
- constraints:

$$Y \in \{0,1\}^{n \times m}, \quad x \in \{0,1\}^n, \quad \mathbf{1}_n^T Y = \mathbf{1}_m, \quad Y \leq [x \; x \; \cdots \; x]$$

Either-Or

we want at least one of the two constraints is satisfied

$$f(x_1, \dots, x_n) \leq 0, \quad g(x_1, \dots, x_n) \leq 0$$

we can introduce $y \in \{0, 1\}$ and a big number M and force

$$f(x_1, \dots, x_n) \leq My, \quad g(x_1, \dots, x_n) \leq M(1 - y)$$

If-Then constraints

we want to ensure that

$$f(x) > 0 \quad \text{implies} \quad g(x) \geq 0$$

- introduce $y \in \{0, 1\}$ and a big number M_f

  $$f(x) \leq M_f \cdot y \quad \text{if } f(x) > 0 \text{ then } y \text{ must be 1}$$

  (if $f(x) \leq 0$ then y can be 0 or 1)
- link the 'then' part: $y = 1 \Rightarrow g(x) \geq 0$

  $$g(x) \geq -M_g(1 - y) \quad \text{if } y = 1 \text{ then } g(x) \geq 0$$

  (if $y = 0$ this constraint does nothing as $-M_g$ is very small)

in conclusion, we formulate

$$f(x) \leq M_f y, \quad g(x) \geq -M_g(1 - y)$$

Fixed-charge problem

scenario: there is a cost associated with performing an activity at a nonzero level that does not depend on the level of activity

a fixed cost of operating machine A when there is an order of product A (but the cost does not depend on the units of A)

modeling:

- introduce binary variable y and let $x \in \mathbf{R}$ be the level/number of units of such activity
- goal: set y to 1 whenever x is nonzero
- use the constraint: $x \leq My$ where M should be set equal to the maximum of x that can attain

Relations between variables

scenario: when at most one of binary variables can be 1

modeling:

- at most one of the variables x_j can be one

$$\mathbf{1}^T x \leq 1$$

- exactly one of the variables x_j should be one

$$\mathbf{1}^T x = 1$$

Disjunctive constraints

scenario: given two constraints $a^T x \geq b$ and $c^T x \geq d$ where $a, c \succeq 0$, we would like to have at least one of the two constraints are satisfied

modeling: define a binary y and impose

$$a^T x \geq yb, \quad c^T x \geq (1-y)d, \quad y \in \{0,1\}$$

general scenario: given m constraints: $a_i^T x \geq b_i$, for $i = 1, \ldots, m$ where $a_i \succeq 0$, we require at least K of them are satisfied

modeling:

$$a_i^T x \geq b_i y_i, \;\; i = 1, 2, \ldots, m, \quad \mathbf{1}^T y \geq K, \;\; y_i \in \{0,1\}$$

Restricted range of values

scenario: aim to restrict x to take values in a set $\{a_1, a_2, \dots, a_m\}$

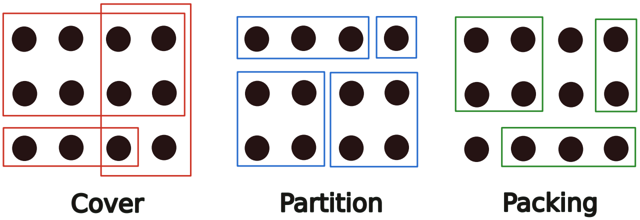
modeling: define $a = (a_1, a_2, \dots, a_m)$ and a binary variable $y \in \mathbf{R}^m$ such that

$$x = a^T y, \quad \mathbf{1}^T y = 1, \quad y_j \in \{0, 1\}$$

Set-covering problems

a set F is a **cover** of M if each member of M must be covered by an acceptable member of set F

objective: minimize a weight (or number of elements) in set F required to cover all elements in M



Modeling examples

Examples: Selection of the dream team (Bertsimas ex 10.2)

The coach of the national basket ball team is faced with the decision of selecting 12 players for an international tournament. He has limited his final selection to 20 players, $p_1, \ldots, p_{20}$. For each player, the coach has collected several statistics that can be summarized as follows. His rebounding average r_i, his assists average a_i, his height h_i, his scoring average s_i, and his overall defense ability d_i. The players have been divided into four broad categories: play makers (PM) $(p_1, \ldots, p_5)$, shooting guards (SG) $(p_4, \ldots, p_{11})$, forwards (F) $(p_9, \ldots, p_{16})$, and centers (C) $(p_{16}, \ldots, p_{20})$. Notice that there are players that can be used in multiple roles (for example players p_4 can be used both as a play maker and a shooting guard). Players p_4, p_8, p_{15}, p_{20} play in the NCAA (college level), while all of the rest play in the NBA (professional level). For balance purposes, the team should consist of at least 3 play makers, 4 shooting guards, 4 forwards, and 3 centers, which implies that some players with dual roles should be selected. In addition, at least 2 players from the NCAA should be selected, while the mean rebounding, assists, scoring average, height, and defense ability should be at least $r_{\min}, a_{\min}, s_{\min}, h_{\min}, d_{\min}$, respectively. The problem is further complicated by the fact that there are compatibility problems among some of the players. Player p_5 has declared that if player p_9 is selected, then he does not want to be in the team. Also, players p_2 and p_{19} can only be selected together as they feel that they are much more effective together. Finally, at most 3 players from the same team should be selected, so that the coach is not accused of favoritism (players p_1, p_7, p_{12}, p_{16} play for the same team). Faced with these difficulties, the coach has decided that he would like to maxmize the scoring average, while satisfying the various constraints. Formulate the problem that the coach is facing as an ILP.

Examples: Selection of the dream team (Bertsimas ex 10.2)

variables $x = (x_1, x_2, \dots, x_n) \in \{0,1\}^n$ is the selection status of the n-player

objective: $s^T x$

constraints:

1. PM ≥ 3, SG ≥ 4, F ≥ 4, C ≥ 3

$$x_1 + \cdots + x_5 \geq 3, \;\; x_4 + \cdots + x_{11} \geq 5, \;\; x_9 + \cdots + x_{16} \geq 4, \;\; x_{16} + \cdots + x_{20} \geq 3$$

2. at least 2 players from NCAA selected: $x_4 + x_8 + x_{15} + x_{20} \geq 2$
3. the mean of all skills must exceed a minimum:

$$r^T x \geq r_{\min}, \;\; a^T x \geq a_{\min}, \;\; h^T x \geq h_{\min}, \;\; s^T x \geq s_{\min}, \;\; d^T x \geq d_{\min}$$

4. if p_9 is selected then p_5 is not in the team: $x_5 \leq (1 - x_9)$
5. player p_2 and p_{19} can only be selected together: $x_2 = x_{19}$
6. at most 3 players from the same team can be selected: $x_1 + x_7 + x_{12} + x_{16} \leq 3$

Example: Either-or constraint (Winston)

Dorian auto is considering manufacturing three types of autos: compact, midsize, and large. The resources required for, and the profits yielded by, each type of car are shown in the Table. Currently, 6,000 tons of steel and 60,000 hours of labor are available. For production of a type of car to be economically feasible, at least 1,000 cars of that type must be produced. Formulate an IP to maximize Dorian's profit.

resource	compact	midsize	large
steel required	1.5 tons	3 tons	5 tons
labor required	30 hours	25 hours	40 hours
profit yielded	2,000 \$	3,000 \$	4,000 \$

Example: Either-or constraint (Winston)

variables:

- $x = (x_1, x_2, x_3) \in \mathbf{Z}^3$: number of compact, midsize, and large cars produced
- $y = (y_1, y_2, y_3) \in \{0,1\}^3$: auxillary variable for either-or constraint

constraints:

1. either $x_i \leq 0$ or $x_i \geq 1,000$ for $i = 1,2,3$

$$\begin{array}{lll} x_1 \leq M_1 y_1, & 1,000 - x_1 \leq M_1(1-y_1), & \text{choose } M_1 = \min\left(\frac{60,000}{30}, \frac{6,000}{1.5}\right) \\ x_2 \leq M_2 y_2, & 1,000 - x_2 \leq M_2(1-y_2), & \text{choose } M_2 = 2,000 \text{ – verify} \\ x_3 \leq M_3 y_3, & 1,000 - x_3 \leq M_3(1-y_3), & \text{choose } M_3 = 1,400 \text{ – verify} \end{array}$$

2. steel and labor constraints

$$1.5x_1 + 3x_2 + 5x_3 \leq 6,000, \quad 30x_1 + 25x_2 + 40x_3 \leq 60,000$$

Example: Fixed-charge IP (Winston)

Gandhi Cloth company manufactures three types of clothing: shirts, shorts, and pants. The manufacture of each type of clothing requires appropriate type of machinery available. The machinery needed to manufacture each type of clothing must be rented at the following rates: shirt machinery, \$200 per week; shorts machiner, \$150 per week; pants machinery, \$100 per week. The manufacture of each type of clothing requires the amounts of cloth and labor shown in the Table. Each week, 150 hours of labor and 160 sq yd of cloth are available. The variable unit cost and selling price for each type of clothing are shown in the Table. Formulate an IP whose solution will maximize Gandhi's weekly profits.

clothing type	labor (hours)	sq yard	sale price (\$)	variable cost (\$)
shirt	3	4	12	6
shorts	2	3	8	4
pants	6	4	15	8

Example: Fixed-charge IP (Winston)

variables:

- $x = (x_1, x_2, x_3) \in \mathbf{Z}^3$: the number of shirts, shorts, and pants
- $y = (y_1, y_2, y_3) \in \{0, 1\}^3$: the status of shirts, shorts, pants being manufactured

constraints:

1. a clothing type is produced if the manufacturing status is on

$$x_1 \leq M_1 y_1, \quad x_2 \leq M_2 y_2, \quad x_3 \leq M_3 y_3, \quad M_1 = 160/4 = 40, M_2 = 53, M_3 = 25$$

2. hours of labor and cloth constraints:

$$3x_1 + 2x_2 + 6x_3 \leq 150, \quad 4x_1 + 3x_2 + 4x_3 \leq 160$$

objective: profit = revenue - cost

$$f(x, y) = (6x_1 + 4x_2 + 7x_3) - (200y_1 - 150y_2 - 100y_3)$$

Example: Facility-location set-covering problem

There are six cities in County A. The county must determine where to build fire stations. The county wants to build the minimum number of fire stations needed to ensure that at least one fire station is within 15 minutes (driving time) of each city. The times (in minutes) required to drive between the cities in County A are shown in the Table. Formulate an IP that will tell County A how many fire stations should be built and where they should be located.

From/To	city 1	city 2	city 3	city 4	city 5	city 6
city 1	0	10	20	30	30	20
city 2	10	0	25	35	20	10
city 3	20	25	0	15	30	20
city 4	30	35	15	0	15	25
city 5	30	20	30	15	0	14
city 6	20	10	20	25	14	0

Example: Facility-location set-covering problem

variable: $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$: status of fire station built

constraints: we create the $n \times n$ incidence matrix A to indicate which locations can reach the given city in 15 minutes or less

example: $a_{11} = a_{12} = 1, a_{21} = a_{22} = a_{26} = 1$

city	within 15 minutes
1	1,2
2	1,2,6
3	3,4
4	3,4,5
5	4,5 6
6	2,5,6

constraints: at least one station is built within 15 minutes from each city

$$x_1 + x_2 \geq 1, \quad x_1 + x_2 + x_6 \geq 1, \quad x_3 + x_3 \geq 1$$

$$x_3 + x_4 + x_5 \geq 1, \quad x_4 + x_5 + x_6 \geq 1, \quad x_2 + x_5 + x_6 \geq 1$$

equivalent: $Ax \geq \mathbf{1}$

objective: minimize $\mathbf{1}^T x$ (the number of stations being built)

Example: Unit commitment problem

given a number of electrical power units n and time horizon T, we determine the startup and shutdown schedule of every unit so that the demand is served and the operating cost is minimized

cost:

1. fixed cost (depend on online status)
2. variable cost (depend on output power)
3. startup cost
4. shutdown cost

constraints:

1. each power unit must operate within a min/max range
2. ramp up limit
3. ramp down limit
4. relation among startup/shutdown/online status
5. output power meets the demand requirement
6. power reserve

Example: Unit commitment problem

variables:

1. startup: $Y \in \{0,1\}^{n\times T}$, $y_{ik}=1$ if unit i is up at the beginning of period k
2. shutdown: $Z \in \{0,1\}^{n\times T}$, $z_{ik}=1$ if unit i is off at the beginning of period k
3. online: $V \in \{0,1\}^{n\times T}$, $v_{ik}=1$ if unit i is online during the period k

cost: coefficients are C^{up} (fixed startup), C^{down} (fixed shutdown), C^{online} (fixed online), and C^{var} (variable)

$$\text{cost} = \sum_{i=1}^{n}\sum_{k=1}^{T}\left(C_{ik}^{\text{up}}y_{ik} + C_{ik}^{\text{down}}z_{ik} + C_{ik}^{\text{online}}v_{ik} + C_{ik}^{\text{var}}p_{ik}\right)$$

Example: Unit commitment problem

constraints:

1. each power unit must operate within a min/max range

$$p_{\min,i} v_{ik} \leq p_{ik} \leq p_{\max,i} v_{ik}, \quad k = 1, \ldots, T, \quad i = 1, 2, \ldots, n$$

2. ramp up limit: $p_{i,t+1} - p_{i,t} \leq \text{ramp up limit}_i, \forall i, \forall t$ where $p_{i,0}$ is given
3. ramp down limit: $p_{i,t} - p_{i,t+1} \leq \text{ramp down limit}_i, \forall i, \forall t$ where $p_{i,0}$ is given
4. relation among startup/shutdown/online status: $y_{ik} - z_{ik} = v_{ik} - v_{i,k-1}$, for $i = 1, \ldots, n, k = 1, \ldots, T$
   (any online unit can be shutdown but not started up, and vice versa)
5. output power meets the demand requirement: $\sum_{i=1}^{n} p_{it} = \text{demand}_t$ for $t = 1, \ldots, T$
6. available power is larger than demand by a reserve

$$\sum_{i=1}^{n} p_{\max,i} v_{it} \geq \text{demand}_t + \text{reserve}_t, \quad t = 1, 2, \ldots, T$$

Relaxation methods

LP relaxation

a LP relaxation of the standard from MILP is

$$\begin{array}{ll} \text{minimize} & c^T x + d^T y \\ \text{subject to} & Ax + By = b \\ & x, y \geq 0 \end{array}$$

- we have dropped the integer constraint
- if MILP has binary variable, its relaxation is $0 \leq x \leq 1$

Facility location problems and its alternative

consider the uncapacitated facility problem on page 13 (left) and its alternative (right) called **aggregate facility problem**

$$\begin{array}{ll}\text{minimize} & \sum_{i=1}^n f_j x_i + \sum_{i=1}^n \sum_{j=1}^m c_{ij} d_j y_{ij} \\ \text{subject to} & y_{ij} \in \{0,1\}, \quad \forall i,j \\ & x_i \in \{0,1\}, \quad \forall i \\ & \sum_{i=1}^n y_{ij} = 1, \quad \forall j \\ & y_{ij} \leq x_i, \quad \forall i,j \end{array} \qquad \begin{array}{ll}\text{minimize} & \sum_{i=1}^n f_j x_i + \sum_{i=1}^n \sum_{j=1}^m c_{ij} d_j y_{ij} \\ \text{subject to} & y_{ij} \in \{0,1\}, \quad \forall i,j \\ & x_i \in \{0,1\}, \quad \forall i \\ & \sum_{i=1}^n y_{ij} = 1, \quad \forall j \\ & \sum_{j=1}^m y_{ij} \leq m x_i, \quad \forall i \end{array}$$

- right: the constraint $\sum_{j=1}^m y_{ij} \leq m x_i$ forces y_{ij} to be 0 whenever $x_i = 0$ but allows y_{ij} to be 1 if $x_i = 1$; therefore, it is equivalent to $y_{ij} \leq x_i$
- however; the right formulation has $m + n$ constraints, while the original has $m + nm$ constraints

Relaxation of facility problem

consider the constraint set that define the relaxations of the original ($\mathcal{C}_1$) and its alternative ($\mathcal{C}_2$)

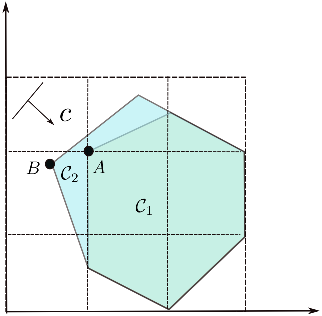
$$\mathcal{C}_1 = \left\{ (x,y) \mid \sum_{i=1}^{n} y_{ij} = 1, \;\; \forall j, \;\; y_{ij} \leq x_i, \;\; \forall i,j, \;\; 0 \leq y_{ij} \leq 1, \;\; 0 \leq x_i \leq 1 \right\}$$

$$\mathcal{C}_2 = \left\{ (x,y) \mid \sum_{i=1}^{n} y_{ij} = 1, \;\; \forall j, \;\; \sum_{j=1}^{m} y_{ij} \leq m x_i, \;\; \forall i, \;\; 0 \leq y_{ij} \leq 1, \;\; 0 \leq x_i \leq 1 \right\}$$

- clearly $\mathcal{C}_1 \subset \mathcal{C}_2$, so $p^\star_{\text{relax},2} \leq p^\star_{\text{relax},1}$
- since the optimal solution to MILP belongs to the constraint set of its relaxation), so $p^\star_{\text{relax},1} \leq p^\star$ (where $p^\star$ is the optimal value of the MILP facility problem)

$$p^\star_{\text{relax},2} \leq p^\star_{\text{relax},1} \leq p^\star$$

Example of feasible set of LP relaxation



- the optimal solution over $\mathcal{C}_1$ is the point A and is integer
- by $p^\star_{\text{relax},2} \leq p^\star_{\text{relax},1} \leq p^\star$, the solution A is indeed the optimal solution to the facility location problem
- the optimal solution over $\mathcal{C}_2$ is point B which is fractional

the relaxation 1 is preferable than relaxation 2 despite the fact that problem 2 has smaller number of constraints

the method that gives sharper bound to $p^\star$ is preferred

Algorithms and softwares

Integer programming methods

- exact algorithms: find an optimal solutions but may take an exponential number of iterations
  - cutting plane
  - branch and bound
  - branch and cut
- approximation algorithms: provide in polynomial time in suboptimal solution together with a bound on the degree of suboptimality
- heuristic algorithms: provide a suboptimal solution but without a guarantee on its quality

Cutting plane methods

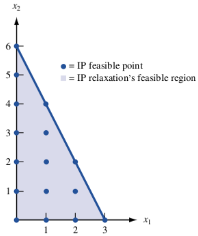
to be included

Branch and bound for a pure IP

we illustrate on a small IP problem

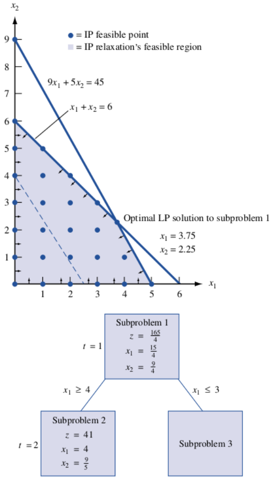
$$\begin{array}{ll} \text{maximize} & 8x_1 + 5x_2 \\ \text{subject to} & x_1 + x_2 \leq 6 \\ & 9x_1 + 5x_2 \leq 45 \\ & x_1, x_2 \geq 0, \quad x_1, x_2 \in \mathbf{Z} \end{array}$$

facts: ✎ assume that the problem is in maximization format

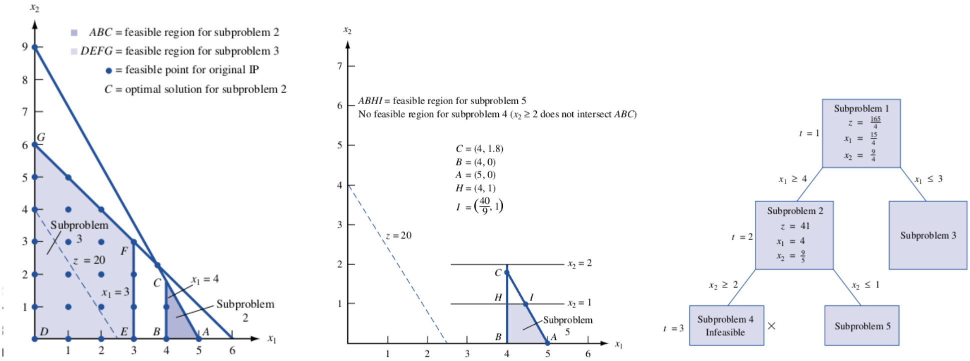


- if the LP relaxation to a pure IP has solution in which all variables are integer then the optimal solution to the LP relaxation is also optimal to the IP
- the optimal value of LP relaxation is an **upper bound** for the original IP
- the z-value of a feasible IP solution provides a **lower bound** of the optimal z-value

Branching

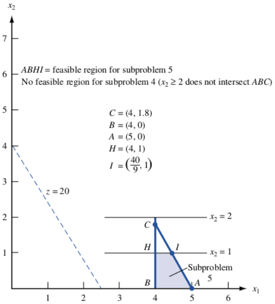


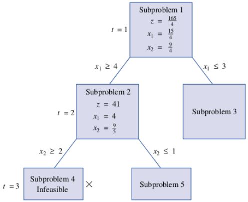
- starts with solving the LP relaxation
- $z = \dfrac{165}{4}, x^\star = (\frac{15}{4}, \frac{9}{4})$
- choose to *partition* the feasible region on x_1 for the IP
- $x_1 \leq 3$ or $x_1 \geq 4$ (since $x_1^\star$ in the relaxation is between 3 and 4
- this step is called **branching** x_1
- create two **subproblems** (LP relaxations) by branching x_1
- subproblem 2 $=$ subproblem 1 $+$ constraint $x_1 \geq 4$
- subproblem 3 $=$ subproblem 1 $+$ constraint $x_1 \leq 3$
- display all subproblems in a tree structure



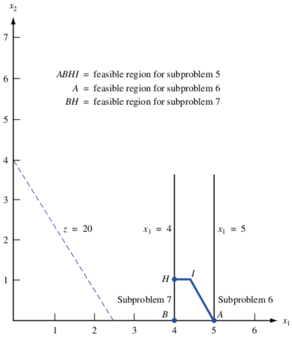
- arbitrarily choose to solve subproblem 2
- point C is optimal in the relaxation with $z = 41, x^\star = (4, \frac{9}{5})$
- the subproblem 2 has $x_2^\star$ as fractional, so we **branch** on x_2
- partition the feasible regions into $x_2 \geq 2$ and $x_2 \leq 1$ creating subproblem 4 and 5

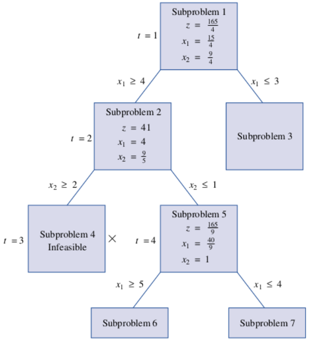
Last-In-First-Out (LIFO)

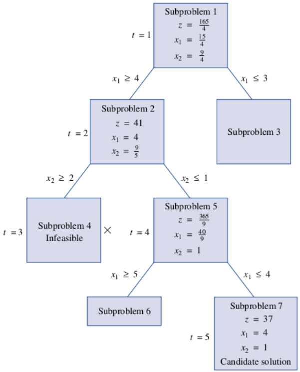




- unsolved problems are subproblem 3,4,5 but we apply LIFO rule
- **LIFO rule:** choose to solve the most **recently created** subproblem (here, 4 or 5)
- we choose to solve subproblem 4 and observe that it is infeasible
- subproblem 4 then provide no useful information and is **fathomed** (place a ×)
- subproblem 5 is solved: $z = \frac{365}{9}, x^\star = (\frac{40}{9}, 1)$ – we branch on x_1
- partition into $x_1 \leq 4$ and $x_1 \geq 5$

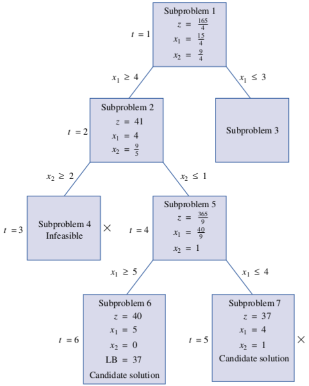






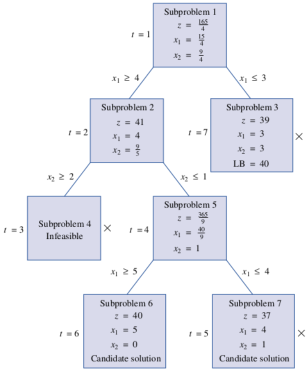
- note that subproblem 6,7 include all integer points that were included in feasible region of subproblem 5
- LIFO rule suggest we should solve #6 or #7 and we choose to solve #7
- optimal solution to subproblem 6 is $z = 37, x^\star = (4, 1)$
- both varaibles are integer so this solution is *feasible* for the original IP

Candidate solutions



- a solution from subproblem in which all variables have integer values is called a **candidate solution**
- since subproblem 7 has a candidate solution, so branching on subproblem 7 will yield no new information about the optimal solution to the IP; hence, this subproblem is fathomed
- the z-value of the candidate solution is a **lower bound** on the optimal z-value for the original IP (keep LB $= 37$)
- the remaining unsolved problems are #3 and #6, but LIFO rules suggests we solve #6

Final branch-and-bound tree



- the optimal solution to subproblem 2 is $z = 40, x^\star = (5, 0)$ ✈ a candidate solution
- since the z-value is greater than the best previous candidate (37), we update LB to 40
- solve the remaining subproblem 3, $z = 39, x^\star = (3, 3)$ but the z-value is smaller than the current LB, so we place $\times$ on $\#3$
- there are no remaining unsolved subproblems
- the only subproblem 6 yield the optimal solution to the IP

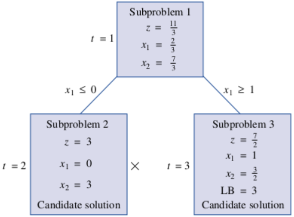
Branch-bound for MILP

✌ just branching only on the variables required to be integers

we illustrate by the following maximization problem

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subjec to} & 5x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 3, \\ & x_1, x_2 \geq 0; \quad x_1 \text{ is integer} \end{array}$$

- the optimal solution of LP relaxation is $z = 11/3, x^\star = (2/3, 7/3)$
- we branch on x_1 and partition the region into $x_1 \leq 0$ and $x_1 \geq 1$



- we choose to solve subproblem 2:
  $z = 3, x^\star = (0, 3)$ ✌ a candidate solution
- now solve subproblem 3: $z = 7/2, x^\star = (1, 3/2)$ ✌ another candidate solution
- since the z-value of subproblem 3 is greater than that of subproblem 2, so subproblem 2 is eliminated

the optimal solution to subproblem 3 is then the optimal solution to the MILP

Modeling softwares

- accept linear programs in standard notation
- recognize problems that can be converted to LPs
- express the problem in the format required by LP solvers
- examples of modeling packages
    - cvx, YALMIP (on MATLAB)
    - cvxpy, cvxopt (on Python)
    - gurobipy (by Gurobi)
    - AMPL

Numerical methods

- simplex (by Dantzig): move along the vertices of polyhedron when the objective is decreasing
- interior-point: move through the interior points of the feasible region
- many libraries/solvers (both commercial and open-source) on the market
  - `intlinprog` in MATLAB
  - Pulp or `scipy.optimize.linprog` in Python
  - Gurobi

References

1. D. Bertsimas and J.N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific, 1997
2. W.L. Winston, *Operations Research: Applications and Algorithms*, Fourth edition, 2004 (Figures of branch-bound methods)
3. I. Griva, S.G. Nash, and A. Sofer, Linear and Nonlinear Optimization, 2nd edition, SIAM, 2009
4. S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge, 2004