4. Estimators

- statistics as estimators
- convergence
- properties of estimators
- sample mean and sample variance

Descriptive statistics

if $x_1, x_2, \ldots, x_N$ are drawn independently from the same population

$\{x_i\}_{i=1,\ldots,N}$ is a **random sample** and said to be independent, identically distribuited (iid)

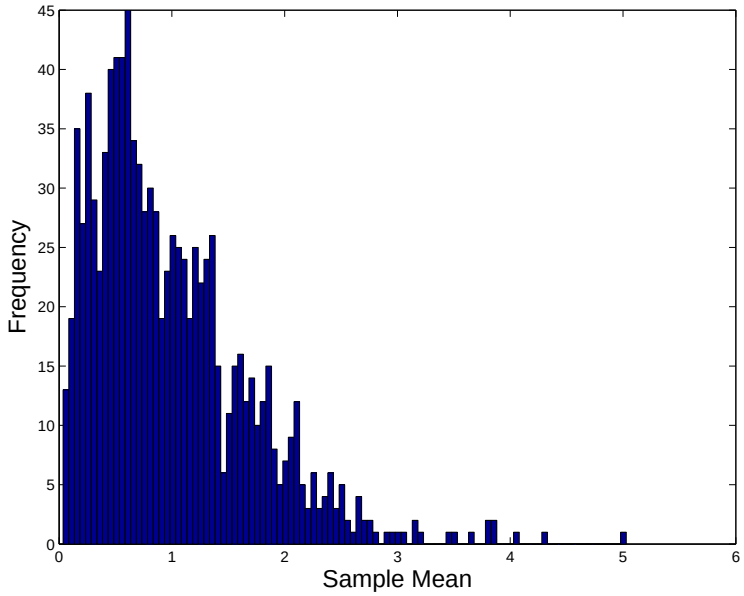
typical summary **statistics** used to describe the sample data

statistic	description	what to describe
mean	$(1/N)\sum_{i=1}^{N} x_i$	central tendency
median	middle ranked observation	central tendency
standard deviation	$\sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^2}{N-1}}$	dispersion
skewness	$\sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^3}{\mathrm{SD}^3(N-1)}}$	asymmetry of pdf
kurtosis	$\sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^4}{\mathrm{SD}^4(N-1)}}$	amount of heavy tails

definition: a **statistic** is any function computed from the data in a sample

- a statistic is a function of random values, so it is also an RV
- the probability distribution of a statistic is called a **sampling distribution**

example: a histogram of 1000 realizations of the sample mean of χ^2_1



the sample mean is calculated on 4 observations

Estimation of parameters

Definition: an **estimator** is a rule for using data to estimate the model parameter

example: to estimate a population mean, one can use *sample mean* or *sample minimum*

- typically, one can compare an estimator with others from their properties
- such properties can be divided into
  - finite sample properties
  - asymptotic properties: when sample size is large

Estimators

- statistics as estimators
- **convergence**
- properties of estimators
- sample mean and sample variance

Convergence of deterministic sequences

Definition: a sequence of *deterministic* numbers $\{a_n : n = 1, 2, \ldots\}$ **converges** to a if

$$\forall \epsilon > 0, \exists N \text{ such that if } n > N \text{ then } |a_n - a| < \epsilon$$

and we write

$$a_n \to a, \quad \text{as } n \to \infty$$

or

$$\lim_{n \to \infty} a_n = a$$

Definition: a sequence a_n is **bounded** if there is some $M < \infty$ such that

$$|a_n| \leq M, \quad \text{for all } n$$

otherwise, we say that a_n is **unbounded**

Convergence in Probability

Definition: a sequence of *random variables* $\{X_n : n = 1, 2, \ldots\}$ **converges in probability** to a random variable X if for all $\epsilon > 0$

$$\lim_{n \to \infty} P(|X_n - X| > \epsilon) = 0$$

and we write

$$X_n \xrightarrow{p} X$$

and say that X is the **probability limit (plim)** of X_n : $\operatorname{plim} X_n = X$

Definition: X_n is **bounded in probability** if for every $\epsilon > 0$, there exists $M_\epsilon < \infty$ and an integer N_ϵ such that

$$P(|X_n| \geq M_\epsilon) < \epsilon, \quad \forall n \geq N_\epsilon$$

example: X_n is a Bernoulli where $P(X_n = 0) = 1 - 1/n$ and $P(X_n = 1) = 1/n$

$x \in \mathbf{R}^{20\times 5}$, contains 20 samples of X_n where $n = (1, 2, 3, 10, 100)$

```
x =
     1     1     1     0     0
     1     0     0     0     0
     1     0     0     0     0
     1     1     0     0     0
     1     0     0     0     0
     1     0     1     0     0
     1     1     0     0     0
     1     0     0     0     0
     1     1     1     0     0
     1     1     0     1     0
     1     0     0     0     0
     1     0     1     0     0
     1     1     0     0     0
     1     0     0     0     0
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     1     0     1     0     0
     1     1     0     0     0
     1     1     0     0     0
     1     1     0     0     0
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x =
     1     0     0     0     0
     1     0     1     0     0
     1     1     0     0     0
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     1     1     0     0     0
     1     0     0     0     0
     1     0     0     0     0
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     1     1     1     0     0
     1     0     0     0     0
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     1     1     0     0     0
     1     0     0     0     0
     1     0     0     1     0
     1     1     0     0     0
     1     0     0     0     0
     1     1     0     0     0
```

X_n converges in probability to 0

Slutsky's Theorem

let $g : \mathbf{R}^m \to \mathbf{R}^p$ be a continuous at some point $x \in \mathbf{R}^m$ and let $\{x_n : 1, 2, \ldots\}$ be a sequence of m-dimensional random vectors

$$\text{if } X_n \xrightarrow{p} X \text{ then } g(X_n) \xrightarrow{p} g(X), \quad \text{as } n \to \infty$$

- in other words, $\operatorname{plim} g(x_n) = g(\operatorname{plim} x_n)$ if g is continuous at x
- plim passes thru nonlinear functions, provided they are continuous
- note that the expectation operator does NOT have this feature

$$\operatorname{plim}(a_n, b_n) = (a, b) \Rightarrow \operatorname{plim}(a_n b_n) = ab \quad \text{but} \quad \mathbf{E}[a_n b_n] \not\Rightarrow \mathbf{E}[a]\mathbf{E}[b]$$

Convergence with Probability One

Definition: a random sequence X_n converges **with probability one** to a random variable X if

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1$$

and denoted by $X_n \overset{as}{\to} X$

- aka **almost sure** or **strong consistency** for X
- almost sure implies convergence in probability (weak consistency for X)

Laws of Large Numbers

theorems for convergence in probability for the sequence of **sample average**

$$\bar{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

where X_i is a random variable

weak law of large numbers:

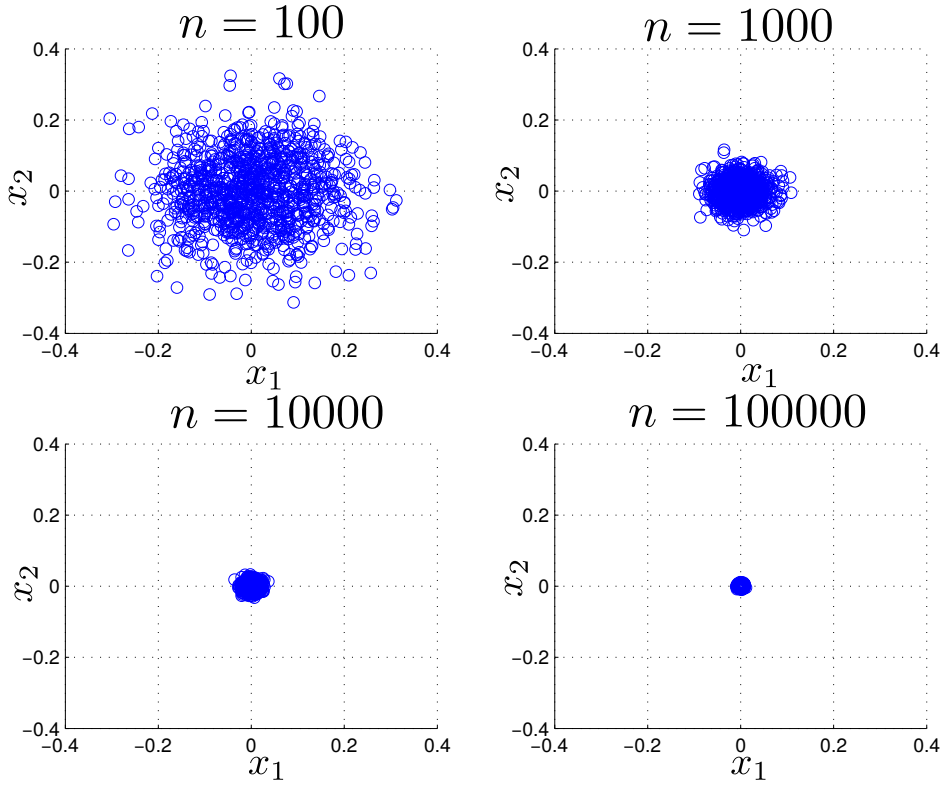
$$\bar{X}_N \xrightarrow{p} \mathbf{E}[\bar{X}_N]$$

if the X_i have common mean μ then this reduces to $\operatorname{plim} \bar{X}_N = \mu$

strong law of large numbers: the convergence is instead almost surely

$$\bar{X}_N \xrightarrow{as} \mathbf{E}[\bar{X}_N]$$

scattergram of 1000 realizations of the sample mean



- X_n is the sample mean and computed from n samples of 2-dimensional Gaussian with zero mean
- as n increases, the probability of that X_n's are concentrated at zero is high

Convergence in Distribution

Definition: a random sequence of X_n **converges in distribution** to the continuous random variable X, denoted by $X_n \xrightarrow{d} X$ if

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \forall x \in \mathbf{R}$$

where F_n is CDF of X_n and F is CDF of X

- example: $t_{n-1} \xrightarrow{d} \mathcal{N}(0,1)$ (t distribution converges to normal)
- it does not imply that X_n converges at all, e.g.,

$$P(X_n = 1) = 1/2 + 1/(n+1), \quad P(X_n = 2) = 1/2 - 1/(n+1)$$

- convergence in probability implies convergence in distribution

$$X_n \xrightarrow{p} X \quad \Longrightarrow \quad X_n \xrightarrow{d} X$$

Continuous Mapping Theorem

if $X_n \xrightarrow{d} X$ and g is a continuous function then

$$g(X_n) \xrightarrow{d} g(X)$$

- an analogue of Slutsky's Theorem for convergence in probability
- useful for determining the asymptotic distribution of test statistics

Transformation Theorem: if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} Y$ then

- $X_n + Y_n \xrightarrow{d} X + Y$
- $X_n Y_n \xrightarrow{d} XY$
- $X_n / Y_n \xrightarrow{d} X/Y$ provided that $P(Y = 0) = 0$

Product Limit Normal Rule

if $X_N \xrightarrow{d} \mathcal{N}(\mu, A)$ and $H_N \xrightarrow{p} H$ where $H \succ 0$ then

$$H_N X_N \xrightarrow{d} \mathcal{N}(H\mu, HAH^T)$$

example of usage: if we have shown that

$$\sqrt{N}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, B)$$

then for any $B_N \succ 0$ that is a consistent estimate for B, we have

$$B_N^{-1/2} \cdot \sqrt{N}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, I)$$

Properties of Estimators

- asymptotic distribution
- unbiased
- consistency (asymptotic properties)
- efficiency (asymptotic properties)

Asymptotic Distribution of Estimators

suppose that

$$\sqrt{N}(\hat{\theta}_N - \theta) \xrightarrow{d} \mathcal{N}(0, P)$$

then we say that

- in *large samples* $\hat{\theta}_N$ is $\sqrt{N}$-**asymptotically normally distributed** with

$$\hat{\theta}_N \sim \mathcal{N}(\theta, N^{-1}P)$$

- the **asymptotic covariance** of $\hat{\theta}_N$ is $N^{-1}P$, denoted by $\mathbf{Avar}[\hat{\theta}_N]$
- $\widehat{\mathbf{Avar}}[\hat{\theta}_N] = N^{-1}\hat{P}$ denotes the **estimate asymptotic variance matrix** of $\hat{\theta}_N$ where $\hat{P}$ is a consistent estimate of P

('in large samples' means N is large enough for $\mathcal{N}(0, P)$ to be a good approximation but not so large that the covariance $N^{-1}P$ goes to zero)

Unbiased Estimators

an estimator $\hat{\theta}$ of θ is said to be **unbiased** if

$$\mathbf{E}[\hat{\theta}] = \mathbf{E}[\theta]$$

example: X_i's are i.i.d with mean μ and variance σ^2

- the expectation of $\bar{X}$ is carried out by

$$\mathbf{E}[\bar{X}] = \mathbf{E}[(1/N)\sum_{i=1}^{N} X_i] = (1/N)\sum_{i=1}^{N} \mathbf{E}[X_i] = (1/N)\sum_{i=1}^{N} \mu = \mu$$

- one can show that the sample variance satisfies $\mathbf{E}[s^2] = \sigma^2$

hence, the sample mean and the sample variance are unbiased estimators of μ and σ^2 respectively

Consistent Estimators

if a sequence of estimators $\hat{\theta}_N$ of θ, where N is the sample size, satisfies

$$\hat{\theta}_N \xrightarrow{p} \theta \quad \text{for all possible values of } \theta$$

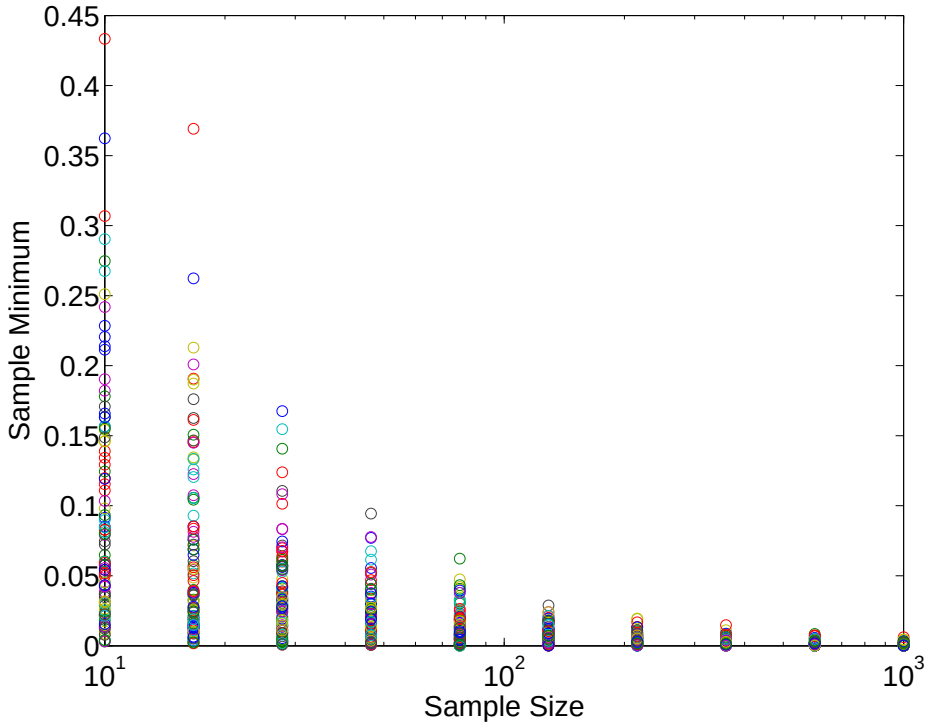
then we say $\hat{\theta}_N$ is a **consistent estimator** of θ

- a consistent estimator converges in probability to the true value
- e.g. the sample mean $\bar{X}_N = (1/N)\sum_i^N X_i$ is a consistent estimator of μ

$$P\left(|\bar{X}_N - \mu)| \geq \epsilon\right) = P\left[\frac{\sqrt{N}|\bar{X}_N - \mu|}{\sigma} \geq \sqrt{N}\epsilon/\sigma\right] = \left(1 - \Phi\left(\frac{\sqrt{N}\epsilon}{\sigma}\right)\right) \to 0$$

as $N \to \infty$ (we have assume X_i's are i.i.d. Gaussian $\mathcal{N}(\mu, \sigma^2)$

example: 100 realizations of the sample minimum of exponential RV



- for each sample size (N), the sample minimum is calculated on N values
- an exponential RV has the minimum at zero
- as N grows, the probability of the sample minimum goes to 0 is approaching 1

Unbiasedness vs Consistency

- unbiasedness needs not imply consistency, $e.g.$, consider i.i.d. sample $X_1, X_2, \ldots, X_n$ of X

$$\hat{\theta} \triangleq X_1, \qquad \mathbf{E}[\hat{\theta}] = \mathbf{E}[X_1] = \mathbf{E}[X] \quad \text{(unbiased)}$$

  but X_1 never converges to any value (not consistent)

- if the sequence does not converge to a value, then it is not consistent, regardless of whether the estimators in the sequence are biased or not
- consistency needs not imply unbiasedness, $e.g.$, $\hat{\theta}_1 \triangleq \hat{\theta}_N + \frac{1}{N}$
  $\hat{\theta}_1$ is still consistent but not unbiased
- consistent estimators are convergent and asymptotically unbiased (so that it converges to the correct value): individual estimators in the sequence may be biased, but the overall sequence still consistent, if the bias converges to zero

Efficient Estimators

a consistent asympotitcally normal estimator $\hat{\theta}_N$ of θ is said to be **asympotically efficient** if it has an asymptotic covariance matrix equal to an **efficiency** lower bound

informally, we will have a theorem stating that for any unbiased estimators, it satisfies

$$\mathbf{Avar}[\hat{\theta}_N] \succeq C$$

where C is an important lower bound, derived from the problem statement/assumptions

therefore, if an estimator of interest happens to satisfy

$$\mathbf{Avar}[\hat{\theta}_N] = C$$

then this estimator is **efficient** (since this is the best we can acheive)

Estimators

- statistics as estimators
- convergence
- properties of estimators
- **sample mean and sample variance**

Sampling statistics

- useful inequalities
- central limit theorem
- sample mean and sample variance

Markov and Chebyshev Inequalities

Markov inequality

let X be a *nonnegative* RV with mean $\mathbf{E}[X]$

$$P(X \geq a) \leq \frac{\mathbf{E}[X]}{a}, \quad a > 0$$

Chebyshev inequality

let X be an RV with mean μ and variance σ^2

$$P\left(|X - \mu| \geq a\right) \leq \frac{\sigma^2}{a^2}$$

Sample mean

let X be an RV with $\mathbf{E}[X] = \mu$ (unknown)

$X_1, X_2, \ldots, X_N$ denote N independent, repeated measurements of X

X_j's are *independent, identically distributed* (i.i.d.) RVs

the **sample mean** of the sequences is used to estimate $\mathbf{E}[X]$:

$$\bar{X} = \frac{1}{N} \sum_{j=1}^{N} X_j$$

two statistical quantities for characterizing the sample mean's properties:

- $\mathbf{E}[\bar{X}]$: we say $\bar{X}$ is unbiased if $\mathbf{E}[\bar{X}] = \mu$
- $\mathbf{var}(\bar{X})$: we examine this value when N is large

the sample mean is an **unbiased estimator** for μ:

$$\mathbf{E}[\bar{X}] = \mathbf{E}\left[\frac{1}{N}\sum_{j=1}^{N} X_j\right] = \frac{1}{N}\sum_{j=1}^{N}\mathbf{E}[X_j] = \mu$$

suppose $\mathbf{var}(X) = \sigma^2$ (true variance)

since X_j's are i.i.d, the variance of $\bar{X}$ is

$$\mathbf{var}(\bar{X}) = \frac{1}{N^2}\sum_{j=1}^{N}\mathbf{var}(X_j) = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$$

hence, the variance of the sample mean approaches zero as the number of samples increases

Weak Law of Large Numbers

let $X_1, X_2, \ldots, X_N$ be a sequence of i.i.d. RVs with finite mean $\mathbf{E}[X] = \mu$ and variance σ^2

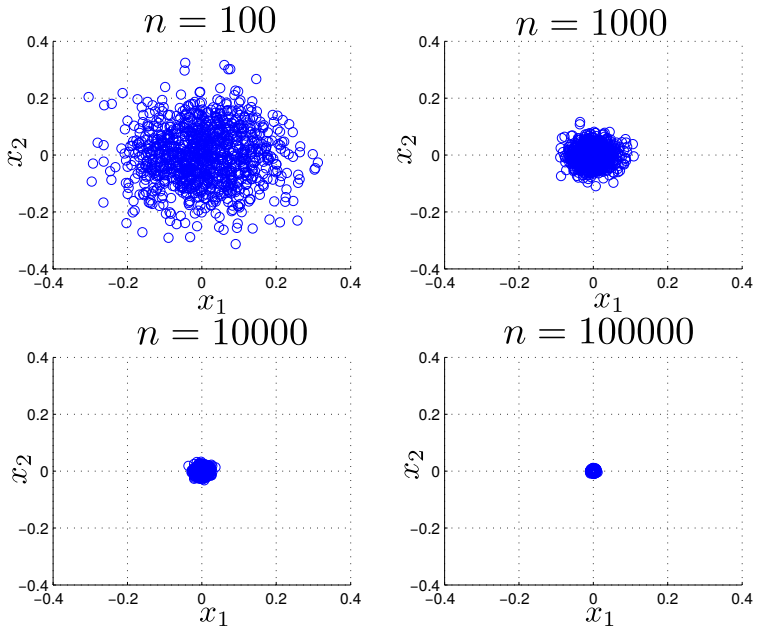
for any $\epsilon > 0$,

$$\lim_{N \to \infty} P[\ |\bar{X} - \mu| < \epsilon\ ] = 1$$

- for large enough N, the sample mean will be close to the true mean with high probability
- *Proof.* apply Chebyshev inequality:

$$P[|\bar{X} - \mu| \geq \epsilon] \leq \frac{\sigma^2}{N\epsilon^2} \quad \Longrightarrow \quad P[|\bar{X} - \mu| < \epsilon] \geq 1 - \frac{\sigma^2}{N\epsilon^2}$$

scattergram of 1000 realizations of the sample mean



- $\bar{X}$'s are computed from 2-dimensional Gaussian with zero mean
- as N increases, the probability of $\bar{X}$'s are concentrated at zero is high

Strong Law of Large Numbers

let $X_1, X_2, \ldots, X_N$ be a sequence of iid RVs with finite mean $\mathbf{E}[X] = \mu$ and finite variance, then

$$P[\lim_{N\to\infty} \bar{X} = \mu] = 1$$

- $\bar{X}_k$ is the sequence of sample mean computed using X_1 through X_k
- with probability 1, every sequence of sample mean calculations will eventually approach and stay close to $\mathbf{E}[X] = \mu$
- the strong law implies the weak law

Central Limit Theorem (CLT)

let $X_1, X_2, \ldots, X_N$ be a sequence of i.i.d. RVs with

finite mean $\mathbf{E}[X] = \mu$ and finite variance σ^2

let S_N be the sum of the first N RVs in the sequences:

$$S_N = X_1 + X_2 + \cdots + X_N$$

and define

$$Z_N = \frac{S_N - N\mu}{\sigma\sqrt{N}}$$

then

$$\lim_{N\to\infty} P(Z_N \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx$$

as N becomes large, the CDF of normalized S_n approaches Gaussian distribution

Proof of Central Limit Theorem

first note that

$$Z_N = \frac{S_N - N\mu}{\sigma\sqrt{N}} = \frac{1}{\sigma\sqrt{N}} \sum_{k=1}^{N} (X_k - \mu)$$

the characteristic function of Z_N is given by

$$\begin{aligned}
\Phi_{Z_N}(\omega) &= \mathbf{E}[e^{\mathrm{i}\omega Z_N}] = \mathbf{E}\left[\exp \frac{\mathrm{i}\omega}{\sigma\sqrt{N}} \sum_{k=1}^{N} (X_k - \mu)\right] \\
&= \mathbf{E}\left[\prod_{k=1}^{N} e^{\mathrm{i}\omega(X_k-\mu)/\sigma\sqrt{N}}\right] \\
&= \left(\mathbf{E}[e^{\mathrm{i}\omega(X-\mu)/\sigma\sqrt{N}}]\right)^N
\end{aligned}$$

(using the fact that X_k's are iid)

expanding the exponential expression gives

$$\begin{aligned}\mathbf{E}[e^{\mathrm{i}\omega(X-\mu)/\sigma\sqrt{N}}] &= \mathbf{E}\left[1+\frac{\mathrm{i}\omega}{\sigma\sqrt{N}}(X-\mu)+\frac{(\mathrm{i}\omega)^2}{2!N\sigma^2}(X-\mu)^2+\ldots\right] \\ &\approx 1-\frac{\omega^2}{2N}\end{aligned}$$

(the higher order term can be neglected as N becomes large)

then we obtain

$$\begin{aligned}\Phi_{Z_N}(\omega) &\rightarrow \left(1-\frac{\omega^2}{2N}\right)^N \\ &\rightarrow e^{-\omega^2/2}, \quad \text{as } N\rightarrow\infty\end{aligned}$$

Multivariate CLT

Lindeberg-Levy Theorem: let $X_1, X_2, \ldots, X_N$ be an i.i.d. sequence of random vectors with $\mathbf{E}[X_i] = \mu$ and $\mathbf{cov}(X_i) = \Sigma$ such that the second moment of each component in X_i is finite

define $\bar{X}_N = (1/N) \sum_{i=1}^{N} X_i$

CLT says that

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - \mathbf{E}[X_i]) = \sqrt{N}(\bar{X}_N - \mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

more conditions involved if X_i's are NOT i.i.d.

Multivariate CLT

Lindeberg-Feller Theorem: let $X_1, X_2, \ldots, X_N$ be samples of random vectors with $\mathbf{E}[X_i] = \mu_i$ and $\mathbf{cov}(X_i) = C_i$ such that all mixed third moments are finite

moreover, assume that for every i

$$\lim_{N\to\infty} \left(\sum_{i=1}^{N} C_i \right)^{-1} C_i = 0, \quad \text{and} \quad C = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} C_i$$

exists and is positive definite

define $\bar{X}_N = (1/N) \sum_{i=1} X_i$ and $\bar{\mu}_N = (1/N) \sum_{i=1} \mu_i$

then CLT says that

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} (X_i - \mu_i) = \sqrt{N}(\bar{X}_N - \bar{\mu}) \xrightarrow{d} \mathcal{N}(0, C)$$

Distribution of $\bar{X}$

let $X_1, \ldots, X_N$ be a sample from a population with mean μ and variance σ^2

let $\bar{X} = (1/N) \sum_{i=1}^{N} X_i$ be the sample mean

- if X_i is **normal**, then $\bar{X}$ is also **normal** with mean μ and variance σ^2/N
- from the central limit theorem, the sample mean is **approximately normal** when N is large where

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{N}}$$

has *approximately* a **standard normal** distribution

Sample Variance

let $X_1, \ldots, X_N$ be a sample from a population with mean μ and variance σ^2

the statistic s^2, defined by

$$s^2 = \frac{\sum_{i=1}^{N}(X_i - \bar{X})^2}{N-1}$$

is called the **sample variance**

- $s = \sqrt{s^2}$ is called the sample standard deviation
- using $(N-1)s^2 = \sum_{i=1}^{N} X_i^2 - N\bar{X}^2$, we have

$$\mathbf{E}[s^2] = \sigma^2 \quad \text{(equal to population variance)}$$

Joint distribution of sample mean and variance

let $X_1, \ldots, X_N$ be a sample from a **normal** popolution, we obtain the identity

$$\sum_{i=1}^{N} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \frac{\sum_{i=1}^{N} (X_i - \bar{X})^2}{\sigma^2} + \left[\frac{\sqrt{N}(\bar{X} - \mu)}{\sigma} \right]^2$$

- LHS is a chi-square of N degrees of freedom
- the second term on RHS is a chi-square with 1 degree of freedom
- the sum of two independent chi-squares with N and M DFs is a chi-square with $N + M$
- it would seem that the first term on RHS is a chi-square with $N - 1$ degree of freedoms

Theorem: if $X_1, \ldots, X_N$ is a sample from a **normal** population with mean μ and variance σ^2

- $\bar{X}$ is normal with mean μ and variance σ^2/N
- $(N-1)s^2/\sigma^2$ is chi-square with $N-1$ degrees of freedom
- $\bar{X}$ and s^2 are **independent**

Corollary: let s be the sample standard deviation

$$\frac{\sqrt{N}(\bar{X}-\mu)}{s} \sim t_{N-1}$$

followed from

$$\frac{\sqrt{N}(\bar{X}-\mu)}{s} = \frac{\sqrt{N}(\bar{X}-\mu)/\sigma}{\sqrt{s^2/\sigma^2}} \triangleq \frac{\text{standard normal}}{\text{chi-square with } N-1 \text{ DF}}$$

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