5. Observer-based Controller Design

- state feedback
- pole-placement design
- regulation and tracking
- state observer
- feedback observer design
- LQR and LQG
State feedback

consider an LTI system

continuous-time

\[ \dot{x} = Ax + Bu, \quad y = Cx + Du \]

discrete-time

\[ x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) \]

Problem: design \( u \) such that the system behaves as desired, for example:

- stabilize the system (if \( A \) is unstable)
- make the output track a reference faster
- make the output small while use less energy of \( u \)
- $r$: reference signal
- $u$: control/actuating signal
- $y$: plant output/controlled signal

**types of control**

- open-loop control: $u$ depends only on $r$ and is independent of $y$
- closed-loop control: $u$ depends on both $r$ and $y$

**state feedback** $u$ is a function of state variables, *i.e.*, $u = f(x, t)$ for some function $f$
Linear state feedback

we study a **constant linear** state feedback:

\[
   u(t) = r(t) - Kx(t)
\]

\(r\) is a reference input and \(K\) is called a **feedback gain**

then the closed-loop system is

\[
\dot{x} = (A - BK)x + Br
\]

eigenvalues of \((A - BK)\) determine the behavior of the closed-loop system
Controllability under a state feedback

**Fact:** \((A - BK, B)\) is controllable if and only if \((A, B)\) is controllability

**Proof:** suppose \((A, B)\) is not controllable, i.e., \(\exists w \neq 0\) such that

\[
w^*A = \lambda w^*, \quad w^*B = 0 \iff w^*(A - BK) = \lambda w^*, \quad w^*B = 0
\]

hence, \((A - BK, B)\) is also not controllable

- the controllability property is *invariant* under any state feedback
- eigenvalues of \((A - BK)\) can be arbitrarily assigned provided that complex conjugate eigenvalues are assigned in pairs
- what about the observability property?
Coordinate transformation

consider a linear transformation $z = T^{-1}x$ where

$$\dot{x} = (A - BK)x + Br$$

the dynamics in the new coordinate is

$$\dot{z} = (\bar{A} - \bar{B}\bar{K})z + \bar{B}r$$

where

$$\bar{A} = T^{-1}AT, \quad \bar{B} = T^{-1}B, \quad \bar{K} = KT$$
Pole-placement design

**Fact:** $\lambda(A - BK)$ cannot be freely reassigned if $(A, B)$ is not controllable

If $(A, B)$ is not controllable then we can put it in uncontrollable form

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

Consider a feedback gain $\bar{K}$ and partition it as

$$\bar{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

Then the closed-loop dynamic matrix is

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} A_{11} - B_1 K_1 & A_{12} - B_1 K_2 \\ 0 & A_{22} \end{bmatrix}$$

$\lambda(A_{22})$ cannot be moved, so uncontrollable modes remain uncontrollable.
Pole placement for single-input systems

**Fact:** $\lambda(A - BK)$ can be freely reassigned if $(A, B)$ is controllable

- change coordinate to the **controller canonical form**

$$
\bar{A} = \begin{bmatrix}
-a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

- assume $\bar{K} = [k_1 \ k_2 \ \cdots \ k_n]$, so

$$
\det(sI - \bar{A} + \bar{B}\bar{K}) = s^n + (a_1 + k_1)s^{n-1} + \cdots + (a_n + k_n)
$$

- choose the closed-loop poles arbitrarily by a suitable choice of $\bar{K}$
- transform $\bar{K}$ back to the new original coordinate
comments:

• drastic change in a desired characteristic polynomial requires a large $K$
• zeros of $C(sI - A)^{-1}B$ are the same as that of $C(sI - A + BK)^{-1}B$

suppose $(A, B, C)$ is in the controller form

$$C(sI - A)^{-1}B = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$C(sI - A + BK)^{-1}B = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n}$$

• the zeros of the transfer function from $r$ to $y$ are not affected by $K$
• state feedback can result in unobservable modes due to cancellation in

$$\frac{C \text{Adj}(sI - A + BK)B}{\det(sI - A + BK)}$$
Regulator problem

consider the state feedback configuration with $r = 0$

Problem: find a state gain $K$ so that $y$ decays to zero at a desired rate

- apply $u = -Kx$, so the closed loop dynamic matrix is $A - BK$
- design $K$ such that $\lambda(A - BK)$ is stable and lies in a desired region
Asymptotic tracking problem

design an overall system so that \( y \to r \) as \( t \to \infty \)

- if \( r(t) = 0 \) then the tracking problem reduces to a regulator problem

- tracking a nonconstant reference is called a **servomechanism** problem
for a tracking problem, we use

\[ u = -K_r r(t) - K x(t) \]

\( K_r \) is a **feedforward gain** and \( K \) is a **feedback gain**

thus we have

\[ \dot{x} = (A - BK)x + BK_r r \]
the transfer function from $r$ to $y$ is

$$\frac{Y(s)}{R(s)} = C(sI - A + BK)^{-1}BK_r$$

assume $(A, B)$ is controllable and $CA^{-1}B \neq 0$

- choose $K$ such that $(A - BK)$ is stable
- choose $K_r$ to make the DC gain from $r$ to $y$:
  $$-C(A - BK)^{-1}BK_r$$
  equal 1

then the closed-loop system can asymptotically track any step reference
Robust tracking and disturbance rejection

A constant disturbance $w$ with unknown magnitude in the model

$$\dot{x} = Ax + Bu + Bw, \quad y = Cx$$

**Objective:** under a presence of

- disturbance $w$
- plant parameter variations (system uncertainties)

design $u$ such that $y$ asymptotically tracks any step reference
**Idea:** add an integrator to the system

**Integrator:** \( \dot{z} = r - y \)

the state-space equation of the augmented system is

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
-C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} r +
\begin{bmatrix}
B \\
0
\end{bmatrix} u +
\begin{bmatrix}
B \\
0
\end{bmatrix} w
\]

\[
y =
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
z
\end{bmatrix}
\]

**Fact:** if \((A, B)\) is controllable and \(C(sI - A)^{-1}B\) has no zero at the origin \((CA^{-1}B \neq 0)\) then

\[
\left(\begin{bmatrix}
A & 0 \\
-C & 0
\end{bmatrix}, \begin{bmatrix}
B \\
0
\end{bmatrix}\right)
\]

is controllable
**Control input:** \( u = -Kx - fz \)

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
A - BK & -Bf \\
-C & 0
\end{bmatrix} \begin{bmatrix}
x \\
z
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} r + \begin{bmatrix}
B \\
0
\end{bmatrix} w, \quad y = \begin{bmatrix}
C \\
0
\end{bmatrix} \begin{bmatrix}
x \\
z
\end{bmatrix}
\]

hence, under the conditions \((A, B)\) controllable and \(CA^{-1}B \neq 0\)

the eigenvalues of the CL system can be freely assigned by \([K \quad f]\)
\[ \tilde{u} = -v - Kx + w \]

\[ G_c(s) = C(sI - A + BK)^{-1}B \]
closed-loop system (under state feedback only)

transfer function: \( G_c(s) = C(sI - A + BK)^{-1}B = \frac{N_c(s)}{D_c(s)} \)

note that

\[
\begin{bmatrix}
    I \\
    -C(sI - A + BK)^{-1}
\end{bmatrix}
\begin{bmatrix}
    0 \\
    1
\end{bmatrix}
\begin{bmatrix}
    sI - A + BK & Bf \\
    C & s
\end{bmatrix}

= \begin{bmatrix}
    sI - A + BK \\
    0
\end{bmatrix}
\begin{bmatrix}
    Bf \\
    s - C(sI - A + BK)^{-1}Bf
\end{bmatrix}

hence,

\[
\det \begin{bmatrix}
    sI - A + BK & Bf \\
    C & s
\end{bmatrix} = \det(sI-A+BK) \cdot (s-C(sI-A+BK)^{-1}Bf)
\]

closed-loop augmented system (with integrator)

characteristic equation: \( \chi_c(s) = sD_c(s) - fN_c(s) \)
by setting $r = 0$, the transfer function from $w$ to $y$ is

$$
\frac{Y(s)}{W(s)} = \frac{sN_c(s)}{\mathcal{X}_c(s)}
$$

hence, if $W(s) = 1/s$ then $Y(s) = \frac{N_c(s)}{\mathcal{X}_c(s)}$

if $\mathcal{X}_c(s)$ contains only stable poles (augmented CL system is stable)

the response of $y$ due to $w$ decays to zero regardless of magnitude of $w$
robust tracking

by setting $w = 0$, the transfer function from $r$ to $y$ is

$$\frac{Y(s)}{R(s)} = \frac{-fN_c(s)}{sD_c(s) - fN_c(s)} = \frac{-fN_c(s)}{\mathcal{X}_c(s)}$$

if $\mathcal{X}_c(s)$ has stable poles, the transfer function from $r$ to $y$ has DC gain=1

the response $y$ tracks the reference even for the presence of parameter variations
Optimal state feedback

Idea:

• a drastic change in pole locations leads to a large feedback gain $K$

• a desired close-loop behavior is satisfied but use a large amount of input

• a trade-off between closed-loop performance and input energy should be considered in the control objective
Linear-quadratic optimal control

consider a *controllable* system

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \]

**LQR problem:** find \( u \) that minimizes

\[
\int_{0}^{\infty} x(t)^*Qx(t) + u(t)^*Ru(t) \, dt
\]

- \( Q \succeq 0 \) determine the cost of state performance
- \( R \succ 0 \) determine the cost of input energy
- \( u \) must stabilize the system, *i.e.*, we must have \( x(t) \to 0 \) as \( t \to \infty \)
- when \( R = 0 \), input \( u \) consists of impulsive inputs that *instantly* drive state to zero, so that optimal cost is zero
- if the system is stable and \( Q = 0 \) then optimal \( u \) is zero
Solution to LQR

introduce the **Algebraic Riccati equation (ARE)**

\[ A^*P + PA - PBR^{-1}B^*P + Q = 0 \]

- ARE is quadratic in \( P \)
- we are interested in a *positive definite* solution \( P \)

the solution of LQR problem is the optimal input \( u \) of the form:

\[ u = -Kx \]

where the optimal feedback gain is

\[ K = R^{-1}B^*P \]

the optimal cost function is \( x_0^*Px_0 \)
Discrete-time LQR problem

consider a \textit{controllable} discrete-time system

\[ x(t + 1) = Ax(t) + Bu(t), \quad x(0) = x_0 \]

**LQR problem:** find \( u \) that minimizes

\[
\sum_{k=0}^{\infty} x(k)^*Qx(k) + u(k)^*Ru(k)
\]

where \( Q \succeq 0 \) and \( R \succ 0 \)
Solution to discrete-time LQR

introduce the **Discrete Algebraic Riccati equation (DARE)**

\[ A^* PA - P - A^* PB (R + B^* PB)^{-1} B^* PA + Q = 0 \]

- DARE is nonlinear in \( P \)
- we are interested in a positive solution \( P \)

the solution of LQR problem is the optimal input \( u \) of the form:

\[ u = -Kx \]

where the optimal feedback gain is

\[ K = R^{-1} B^* P \]

the optimal cost function is \( x_0^* Px_0 \)
State observer

Idea:

- a state feedback requires the availability of all state variables
- if state variables cannot be acquired, we must design a state estimator

Consider a state equation

\[
\dot{x} = Ax + Bu, \quad y = Cx
\]

Simple scheme: imitate the original system

\[
\dot{x} = A\hat{x} + Bu
\]

- if \((A, C)\) is observable, then \(x(0)\) can be estimated
- initialize \(\hat{x}\) by using \(x(0)\) then \(x(t) = \hat{x}(t)\) for all \(t \geq T\) (for some \(T\))
open-loop observer

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

drawbacks:

- the initial state must be estimated each time we use the observer
- if \( A \) is unstable then the error between \( x \) and \( \hat{x} \) grows with time

open-loop observer is not satisfactory in general
closed-loop state observer

we modify the state observer as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

add a correction term and design a proper gain $L$
**observer gain:** how to choose $L$?

Define the error between the actual state and the estimated state

$$e = x - \hat{x}$$

then the dynamic of $e$ is

$$\dot{e} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A - LC)\dot{\hat{x}} - Bu - L(Cx)$$

$$\quad = (A - LC)x - (A - LC)\hat{x} = (A - LC)e$$

- if $(A - LC)$ is stable then $e \to 0$, or $\hat{x}$ approach $x$ eventually
- even if there is an initial large error $e(0)$, $e(t)$ still goes zero as $t \to \infty$ if $(A - LC)$ is stable
- no need to compute the initial estimate $\hat{x}(0)$ perfectly
**Observer design**

**Fact:** eigenvalues of $A - LC$ can be freely assigned iff $(A, C)$ is observable

- change coordinate to the **observer canonical form**

$$\bar{A} = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1} & 0 & 0 & 1 & 0 \\ -a_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \bar{C} = [1 \ 0 \ 0 \ \cdots \ 0]$$

- assume $\bar{L} = [l_1 \ l_2 \ \cdots \ l_n]^T$, so

$$\det(sI - \bar{A} + \bar{L}\bar{C}) = s^n + (a_1 + l_1)s^{n-1} + \cdots + (a_n + l_n)$$

- choose the closed-loop poles arbitrarily by a suitable choice of $\bar{L}$
- transform $\bar{L}$ back to the new original coordinate
Remarks:

- observer design procedure can be obtained from the duality theorem:
  \[(A, C)\] is observable if and only if \[(A^*, C^*)\] is controllable

- eigenvalues of \[(A^* - C^K)\] can be freely assigned by \(K\) if \[(A^*, C^*)\] controllable

- eigenvalues of \[(A^* - C^K)\] are the same as that of \[(A - K^*C)\]

- we can pick \(L = K^*\)

- designing an observer gain is equivalent to designing a state feedback gain for the dual system
Feedback from estimated states

when \( x \) is not available, we apply a state feedback from \( \hat{x} \)

\[
u = r - K\hat{x}
\]
	his is called an observer-based controller

we have to answer the following questions

• is the closed-loop system stable?

• using \( u = -K\hat{x} \) gives the same set of eigenvalues as using \( u = -Kx \) ?

• what is the effect of the observer on the transfer function from \( r \) to \( y \) ?
the state equation of the closed loop system

\[
\begin{align*}
\dot{x} &= Ax - BK\hat{x} + Br \\
\hat{x} &= A\hat{x} + Bu + L(y - C\hat{x}) = (A - LC')\hat{x} + B(r - K\hat{x}) + LCx
\end{align*}
\]
Separation property

the state equation in a vector form is

\[
\begin{bmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{bmatrix} =
\begin{bmatrix}
A & -BK \\
LC & A - LC - BK
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} +
\begin{bmatrix}
B \\
B
\end{bmatrix} r,
\quad y =
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix}
\]

change of coordinate: define \( e = x - \hat{x} \)

\[
\begin{bmatrix}
x \\
e
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
I & -I
\end{bmatrix}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix} \quad \triangleq \quad T^{-1}
\begin{bmatrix}
x \\
\hat{x}
\end{bmatrix}
\]

hence, in the new coordinate the state equation is

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} =
\begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix} +
\begin{bmatrix}
B \\
0
\end{bmatrix} r,
\quad y =
\begin{bmatrix}
C & 0
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

• closed-loop eigenvalues are \( \{\text{eig}(A - BK)\} \cup \{\text{eig}(A - LC)\} \)

• designs of state feedback and observer can be done \textit{independently}
Transfer function of the system with observer

the state equation

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r, \quad y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}
\]

is of the **uncontrollable** form

hence, the transfer function equals that of the reduced equation

\[
\dot{x} = (A - BK)x + Br, \quad y = Cx
\]

or the transfer function from \( r \) to \( y \) is

\[
G(s) = C(sI - A + BK)^{-1}B
\]

same as the transfer function of the orginal state feedback system *without* using an observer

Observer-based Controller Design 5-35
what are good choices of $K$ and $L$? no simple answer

some ideas:

- **LQG control**: $K$ and $L$ are chosen to optimize a quadratic objectives and we need to solve two decoupled Riccati equations.

- **$\mathcal{H}_\infty$ control**: $K$ and $L$ are chosen to optimize an $\mathcal{L}$-induced norm of the closed-loop system. need to solve two coupled Riccati equations.

- **$\mathcal{L}_1$ control**: $K$ and $L$ are chosen to optimize a peak-amplitude of regulated output. need to solve optimization problem (LP).

- **multi-objectives, e.g., mixed LQG/$\mathcal{H}_\infty$**.
Summary

• eigenvalues of \((A - BK)\) can be freely reassigned iff \((A, B)\) is controllable

• optimal LQR control input is a constant state feedback computed via ARE

• feedback observer design is equivalent to state feedback design on the dual system

• observer-based controller combines observer and state-feedback designs
References

Lecture note on

*Observer-based Controller Design*, D. Banjerdpungchai, EE635, Chulalongkorn University

Chapter 8 in