4. Minimal realization

- minimal realization
- Popov-Belevitch-Hautus (PBH) tests

Uncontrollable/Unobservable systems

find a state-space description of

$$H(s) = \frac{1}{s+1}$$

one example is a scalar system that is both controllable and observable:

$$\dot{x} = -x + u, \quad y = x$$

or a second-order system that is controllable but unobservable:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & -1 \end{bmatrix} x$$

or a second-order system that is observable but uncontrollable:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Minimal realization

uncontrollable or unobservable systems have common roots between

$$C\text{adj}(sI - A)B \quad \text{and} \quad \det(sI - A)$$

Results

- some eigenvalues of A do not appear in $H(s)$
- $H(s)$ has a lower order than the dimension of the state space
- such state-space is called *non-minimal*

Definition: $\{A, B, C\}$ is a *mininal* realization if there can be no other realization $\{\bar{A}, \bar{B}, \bar{C}\}$ with $\bar{A}$ of smaller dimension than A

Theorem a realization $\{A, B, C\}$ is minimal if and only if

$$a(s) \triangleq \det(sI - A) \quad \text{and} \quad b(s) \triangleq C\,\mathrm{adj}(sI - A)B$$

are relatively coprime

Proof. suppose $\{A, B, C\}$ is minimal but $b(s)/a(s)$ is reducible

then we can find a realization with a lower-dimensional state space of the reduced transfer function, which is a contradiction

to prove the converse, assume that $\{A, B, C\}$ is not minimal even though $b(s)/a(s)$ is irreducible

then any minimal realization of $H(s)$ will have a transfer function with denominator of degree less than the dimension of A

hence, $b(s)/a(s)$ could not have been irreducible

Theorem a realization $\{A, B, C\}$ is minimal if and only if (A, B) is controllable and (A, C) is observable

Proof.

- *sufficiency part.* since we have shown if (A, B) is uncontrollable or (A, C) is unobservable then there exists $\{A_{11}, B_1, C_1\}$ that gives the same $H(s)$ but with a lower dimension
- *necessity part.* we will prove by contradiction $i.e.$, suppose (A, B, C) is controllable and observable but $\{A, B, C\}$ is not minimal

suppose $\{A, B, C\}$ and $\{\bar{A}, \bar{B}, \bar{C}\}$ have the same $H(s)$ where $A \in \mathbf{R}^{n \times n}$ and $\bar{A} \in \mathbf{R}^{r \times r}$, $r < n$

the impulse responses of the two realization must be equivalent, $i.e.$,

$$CA^kB = \bar{C}\bar{A}^k\bar{B}, \quad k = 0, 1, \dots$$

or equivalently,

$$\mathcal{O}\mathcal{C} = \bar{\mathcal{O}}_n \bar{\mathcal{C}}_n$$

where $\bar{\mathcal{C}}_n$ is defined by

$$\bar{\mathcal{C}}_n \triangleq \begin{bmatrix} \bar{B} & \bar{A}\bar{B} & \dots & \bar{A}^{n-1}\bar{B} \end{bmatrix}$$

and defined similarly for $\bar{\mathcal{O}}_n$

since $\bar{\mathcal{O}}_n$ and $\bar{\mathcal{C}}_n$ has size $n \times r$ and $r \times n$, respectively, the matrix $\bar{\mathcal{O}}_n \bar{\mathcal{C}}_n$ has rank at most r

however, (A, B, C) is controllable and observable, then $\mathbf{rank}(\mathcal{O}) = n$ and $\mathbf{rank}(\mathcal{C}) = n$ which implies $\mathbf{rank}(\mathcal{O}\mathcal{C}) = n$

then $\bar{\mathcal{O}}_n \bar{\mathcal{C}}_n$ must also have rank n, which is a contradiction

PBH eigenvector tests

Controllability: A pair (A, B) is controllable if and only if there is no vector $w \neq 0$ and $\lambda \in \mathbb{C}$ such that

$$w^* A = \lambda w^*, \quad \text{and} \quad w^* B = 0$$

$i.e.$, there is no left eigenvector of A that is orthogonal to the columns of B

Observability: A pair (A, C) is observable if and only if there is no vector $v \neq 0$ and $\lambda \in \mathbb{C}$ such that

$$Av = \lambda v, \quad \text{and} \quad Cv = 0$$

$i.e.$, there is no eigenvector of A that is orthogonal to the rows of C

Proof of controllability test

- *sufficiency part.* we show that if $\exists w \neq 0$, $w^* A = \lambda w^*$ and $w^* B = 0$ then (A, B) is uncontrollable

$$w^* B = 0 \Rightarrow w^* AB = \lambda w^* B = 0, \quad \cdots \quad \Rightarrow w^* A^{n-1} B = 0$$

hence, $w^* \mathcal{C} = 0$ or $\mathcal{N}(\mathcal{C}^*) \neq \{0\}$, *i.e.*, (A, B) is uncontrollable

- *necessity part.* if (A, B) is uncontrollable, we can transform the system into the uncontrollable form

$$T^{-1} A T = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{bmatrix}, \quad T^{-1} B = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}$$

let w_2 be a left eigenvector of $\overline{A}_{22}$ then we can show that

$$\begin{bmatrix} 0 & w_2^* \end{bmatrix} T^{-1} \cdot A = \lambda \begin{bmatrix} 0 & w_2^* \end{bmatrix} T^{-1}, \quad \text{and} \quad T^{-1} B = 0$$

(we have found a left eigenvector of A that is orthogonal to B)

PBH rank tests

let $A \in \mathbf{R}^{n\times n}$

Controllability: (A, B) is controllable if and only if

$$\mathbf{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}$$

Observability: (A, C) is observable if and only if

$$\mathbf{rank}\begin{bmatrix} C \\ sI - A \end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}$$

the rank must be n even when s is an *eigenvalue* of A

Proof of controllability test

if $s \neq \lambda(A)$ then $\mathbf{rank}(sI - A) = n$ and so is $\mathbf{rank}\begin{bmatrix} sI - A & B \end{bmatrix}$

therefore, we can just prove only when $s = \lambda$, an eigenvalue of A

- assume (A, B) controllable but $\mathbf{rank}\begin{bmatrix} sI - A & B \end{bmatrix} < n$
- there must exist $w \neq 0$ such that $w^* \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0$
- hence, $w^*(\lambda I - A) = 0$ and $w^* B = 0$
- by the PBH eigenvector test, this implies w is a left eigenvector of A that is orthogonal to B
- so (A, B) must be uncontrollable, which is a contraction

PBH eigenvector test implies that if (A, B) is uncontrollable then

$$\exists w \neq 0, \quad w^* A = \lambda w^*, \quad \text{and} \quad w^* B = 0$$

hence, the dynamic of a special linear combination of $x(t)$, given by

$$\frac{dw^* x(t)}{dt} = w^*(Ax(t) + Bu(t)) = \lambda w^* x(t)$$

clearly does not depend on $u(t)$

similarly, if (A, C) is unobservable, $i.e.$,

$$\exists v \neq 0, \quad Av = \lambda v, \quad \text{and} \quad Cv = 0$$

then given $x(0) = v$, we have

$$x(t) = e^{\lambda t} v, \quad y = Cx(t) = e^{\lambda t} Cv = 0$$

the mode corresponds to λ is unobservable

References

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