4. Minimal realization

- minimal realization

- Popov-Belevitch-Hautus (PBH) tests
Uncontrollable/Unobservable systems

find a state-space description of

\[ H(s) = \frac{1}{s + 1} \]

one example is a scalar system that is both controllable and observable:

\[ \dot{x} = -x + u, \quad y = x \]

or a second-order system that is controllable but unobservable:

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
u
\end{bmatrix}, \quad y = \begin{bmatrix} 1 & -1 \end{bmatrix} x
\]

or a second-order system that is observable but uncontrollable:

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
u
\end{bmatrix}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]
Minimal realization

uncontrollable or unobservable systems have common roots between

\[ C \text{adj}(sI - A)B \quad \text{and} \quad \det(sI - A) \]

Results

• some eigenvalues of \( A \) do not appear in \( H(s) \)
• \( H(s) \) has a lower order than the dimension of the state space
• such state-space is called non-minimal

Definition: \( \{A, B, C\} \) is a minimal realization if there can be no other realization \( \{\bar{A}, \bar{B}, \bar{C}\} \) with \( \bar{A} \) of smaller dimension than \( A \)
**Theorem** a realization \( \{A, B, C\} \) is minimal if and only if

\[
a(s) \triangleq \det(sI - A) \quad \text{and} \quad b(s) \triangleq C \adj(sI - A)B
\]

are relatively coprime

**Proof.** suppose \( \{A, B, C\} \) is minimal but \( b(s)/a(s) \) is reducible

then we can find a realization with a lower-dimensional state space of the reduced transfer function, which is a contradiction

to prove the converse, assume that \( \{A, B, C\} \) is not minimal even though \( b(s)/a(s) \) is irreducible

then any minimal realization of \( H(s) \) will have a transfer function with denominator of degree less than the dimension of \( A \)

hence, \( b(s)/a(s) \) could not have been irreducible
Theorem a realization \( \{A, B, C\} \) is minimal if and only if \((A, B)\) is controllable and \((A, C)\) is observable.

Proof.

- **sufficiency part.** Since we have shown if \((A, B)\) is uncontrollable or \((A, C)\) is unobservable then there exists \(\{A_{11}, B_1, C_1\}\) that gives the same \(H(s)\) but with a lower dimension.

- **necessity part.** We will prove by contradiction \(i.e.,\) suppose \((A, B, C)\) is controllable and observable but \(\{A, B, C\}\) is not minimal.

Suppose \(\{A, B, C\}\) and \(\{\bar{A}, \bar{B}, \bar{C}\}\) have the same \(H(s)\) where \(A \in \mathbb{R}^{n \times n}\) and \(\bar{A} \in \mathbb{R}^{r \times r}, r < n\).

The impulse responses of the two realization must be equivalent, \(i.e.,\)

\[ CA^kB = \bar{C}\bar{A}^k\bar{B}, \quad k = 0, 1, \ldots \]
or equivalently,

\[ OC = \bar{O}_n \bar{C}_n \]

where \( \bar{C}_n \) is defined by

\[ \bar{C}_n \triangleq [ \bar{B} \bar{A} \bar{A} \ldots \bar{A}^{n-1} \bar{B} ] \]

and defined similarly for \( \bar{O}_n \)

since \( \bar{O}_n \) and \( \bar{C}_n \) has size \( n \times r \) and \( r \times n \), respectively, the matrix \( \bar{O}_n \bar{C}_n \) has rank at most \( r \)

however, \( (A, B, C) \) is controllable and observable, then \( \text{rank}(O) = n \) and \( \text{rank}(C) = n \) which implies \( \text{rank}(OC) = n \)

then \( \bar{O}_n \bar{C}_n \) must also have rank \( n \), which is a contradiction
**PBH eigenvector tests**

**Controllability:** A pair \((A, B)\) is controllable if and only if there is no vector \(w \neq 0\) and \(\lambda \in \mathbb{C}\) such that

\[
w^* A = \lambda w^*, \quad \text{and} \quad w^* B = 0
\]

\(i.e.,\) there is no left eigenvector of \(A\) that is orthogonal to the columns of \(B\)

**Observability:** A pair \((A, C)\) is observable if and only if there is no vector \(v \neq 0\) and \(\lambda \in \mathbb{C}\) such that

\[
Av = \lambda v, \quad \text{and} \quad Cv = 0
\]

\(i.e.,\) there is no eigenvector of \(A\) that is orthogonal to the rows of \(C\)
Proof of controllability test

- **sufficiency part.** we show that if \( \exists w \neq 0, w^* A = \lambda w^* \) and \( w^* B = 0 \) then \((A, B)\) is uncontrollable

  \[
  w^* B = 0 \implies w^* A B = \lambda w^* B = 0, \quad \cdots \implies w^* A^{n-1} B = 0
  \]

  hence, \( w^* C = 0 \) or \( \mathcal{N}(C^*) \neq \{0\} \), i.e., \((A, B)\) is uncontrollable

- **necessity part.** if \((A, B)\) is uncontrollable, we can transform the system into the uncontrollable form

  \[
  T^{-1} A T = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{bmatrix}, \quad T^{-1} B = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}
  \]

  let \( w_2 \) be a left eigenvector of \( \overline{A}_{22} \) then we can show that

  \[
  \begin{bmatrix} 0 & w_2^* \end{bmatrix} T^{-1} \cdot A = \lambda \begin{bmatrix} 0 & w_2^* \end{bmatrix} T^{-1}, \quad \text{and} \quad T^{-1} B = 0
  \]

  (we have found a left eigenvector of \( A \) that is orthogonal to \( B \))
let $A \in \mathbb{R}^{n \times n}$

**Controllability:** $(A, B)$ is controllable if and only if

$$\text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}$$

**Observability:** $(A, C)$ is observable if and only if

$$\text{rank} \begin{bmatrix} C \\ sI - A \end{bmatrix} = n \quad \text{for all } s \in \mathbb{C}$$

the rank must be $n$ even when $s$ is an *eigenvalue* of $A$
Proof of controllability test

if \( s \neq \lambda(A) \) then \( \text{rank}(sI - A) = n \) and so is \( \text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} \)

therefore, we can just prove only when \( s = \lambda \), an eigenvalue of \( A \)

- assume \( (A, B) \) controllable but \( \text{rank} \begin{bmatrix} sI - A & B \end{bmatrix} < n \)
- there must exist \( w \neq 0 \) such that \( w^* \begin{bmatrix} \lambda I - A & B \end{bmatrix} = 0 \)
- hence, \( w^*(\lambda I - A) = 0 \) and \( w^*B = 0 \)
- by the PBH eigenvector test, this implies \( w \) is a left eigenvector of \( A \) that is orthogonal to \( B \)
- so \( (A, B) \) must be uncontrollable, which is a contraction
PBH eigenvector test implies that if \((A, B)\) is uncontrollable then

\[
\exists w \neq 0, \quad w^* A = \lambda w^*, \quad \text{and} \quad w^* B = 0
\]

hence, the dynamic of a special linear combination of \(x(t)\), given by

\[
\frac{d w^* x(t)}{dt} = w^* (Ax(t) + Bu(t)) = \lambda w^* x(t)
\]

clearly does not depend on \(u(t)\)

similarly, if \((A, C)\) is unobservable, \(i.e.,\)

\[
\exists v \neq 0, \quad Av = \lambda v, \quad \text{and} \quad Cv = 0
\]

then given \(x(0) = v\), we have

\[
x(t) = e^{\lambda t} v, \quad y = C x(t) = e^{\lambda t} Cv = 0
\]

the mode corresponds to \(\lambda\) is unobservable
References

Chapter 2 in


Chapter 5 in