6. Linear Quadratic Regulator Control

- algebraic Riccati Equation (ARE)
- infinite-time LQR (continuous)
- Hamiltonian matrix
- gain margin of LQR

Algebraic Riccati Equation (ARE)

given $R \succ 0$ and $Q \succeq 0$ and a square matrix A

we solve P from

$$PA + A^*P - PBR^{-1}B^*P + Q = 0$$

- ARE may have more than one solution
- P can be non-symmetric, indefinite, negative definite or positive definite
- we are interested in a **non-negative** solution
- sometimes ARE is called **steady-state Riccati equation (SSRE)**

Positive definite solution

assume $P \succeq 0$, we can imply $P \succ 0$ if *any* of the following is true:

1. $Q \succ 0$
2. $Q \succeq 0$ and (A, Q) observable

Proof 1 easy to check that $\mathcal{N}(P) \subseteq \mathcal{N}(Q)$

then if $\mathcal{N}(Q) = \{0\}$, so is $\mathcal{N}(P)$

to show this, we can see that for any x,

$$\langle PAx, x\rangle + \langle A^*Px, x\rangle - \langle PBR^{-1}B^*Px, x, \rangle + \langle Qx, x\rangle = 0$$

hence, if $Px = 0$ then $Qx = 0$

Proof 2 define $\mathcal{A} = A - BR^{-1}B^*P$ and we can write ARE as

$$P\mathcal{A} + \mathcal{A}^*P + PBR^{-1}B^*P + Q = 0$$

by adding and substracting $PBR^{-1}B^*P$

- take an inner product with $e^{\mathcal{A}t}z$

$$\frac{d}{dt}\langle Pe^{\mathcal{A}t}z, e^{\mathcal{A}t}z\rangle = -\|R^{-1/2}B^*Pe^{\mathcal{A}t}z\|^2 - \langle Qe^{\mathcal{A}t}z, e^{\mathcal{A}t}z\rangle$$

- integrate from 0 to t on both sides

$$\langle Pe^{\mathcal{A}t}z, e^{\mathcal{A}t}z\rangle = \langle Pz, z\rangle - \int_0^t \|R^{-1/2}B^*Pe^{\mathcal{A}\tau}z\|^2 + \langle Qe^{\mathcal{A}\tau}z, e^{\mathcal{A}\tau}z\rangle d\tau$$

- hence $0 \leq \langle Pe^{\mathcal{A}t}z, e^{\mathcal{A}t}z\rangle \leq \langle Pz, z\rangle$ and

$$\text{if } \exists z \neq 0 \text{ s.t. } \quad Pz = 0 \quad \Longrightarrow \quad Pe^{\mathcal{A}t}z = 0$$

- then we can conclude

$$\begin{aligned} \forall z \in \mathcal{N}(P) \;\; Pz = 0 \quad &\implies \quad \mathcal{A}z = Az \\ &\implies \quad e^{\mathcal{A}t}z = e^{At}z \\ &\implies \quad Pe^{At}z = Pe^{\mathcal{A}t}z = 0 \end{aligned}$$

- this implies $e^{At}z \in \mathcal{N}(P)$, *e.g.*, $\mathcal{N}(P)$ is invariant under e^{At}

- since $\mathcal{N}(P) \subseteq \mathcal{N}(Q)$ then

$$Pe^{\mathcal{A}t}z = 0 \quad \implies \quad Qe^{At}z = 0$$

  which contradicts to that (A, Q) is observable

- this also shows

$$\mathcal{N}(P) \subseteq \mathcal{M}_{uo}(A, Q) \subseteq \mathcal{N}(Q)$$

Stability of $\mathcal{A}$

define $\mathcal{A} = A - BR^{-1}B^*P$ and assume $P \succeq 0$ is a solution to ARE

Fact: $\mathcal{A}$ is stable if *either* one of the following is true

1. $Q \succ 0$
2. $Q \succeq 0$ and (A, Q) observable

ARE can be rewritten as

$$P\mathcal{A} + \mathcal{A}^*P + PBR^{-1}B^*P + Q = 0$$

suppose x is an eigenvector of $\mathcal{A}$, $i.e.$, $\mathcal{A}x = \lambda x$

multiplying x with ARE and taking an inner product with x give

$$2\text{Re }\lambda\langle Px, x\rangle = -\|R^{-1/2}B^*Px\|^2 - \langle Qx, x\rangle$$

Proof 1 if $Q \succ 0$ then $P \succ 0$ (page 6-3) and hence, Re $(\lambda) \leq 0$

Proof 2 If Re $\lambda = 0$ or $\lambda = i\omega$, then

$$B^* P x = 0 \quad \text{and} \quad Qx = 0$$

$B^* P x = 0$ implies $\mathcal{A}x = Ax = i\omega x$ and hence

$$Qe^{At}x = Qe^{i\omega t}x = e^{i\omega t}Qx = 0$$

which contradicts to that (A, Q) is observable

conclusion: $\mathcal{A}$ is stable if we use the **positive** solution P

rearrange the ARE as a Lyapunov equation for the closed-loop

$$P\mathcal{A} + \mathcal{A}^* P + K^* R K + Q = 0$$

where $K = -R^{-1} B^* P$

Converse theorem

assume A is stable then

$$\mathcal{M}_{uo}(A, Q) = \mathcal{N}(P)$$

in other words, for a stable A, observability of (A, Q) implies $P \succ 0$

Proof multiply ARE with $e^{At}z$ and taking an inner product with $e^{At}z$

$$\frac{d}{dt}\langle Pe^{At}z, e^{At}z\rangle = \|R^{-1/2}B^*Pe^{At}z\|^2 - \langle Qe^{At}z, e^{At}z\rangle$$

integrate from 0 to t on both sides

$$\langle Pe^{At}z, e^{At}z\rangle - \langle Pz, z\rangle = \int_0^t \|R^{-1/2}B^*Pe^{A\tau}z\|^2 d\tau - \int_0^t \langle Qe^{A\tau}z, e^{A\tau}z\rangle d\tau$$

let $t \to \infty$ and hence $e^{At} \to 0$

$$0 \leq \langle Pz, z \rangle \leq \int_0^\infty \langle Qe^{A\tau}z, e^{A\tau}z \rangle d\tau$$

for all $t \geq 0$, if $Qe^{At}z = 0$ then $Pz = 0$

this means

$$\mathcal{M}_{uo}(A, Q) \subseteq \mathcal{N}(P)$$

in combination with the result in page 6-5 that

$$\mathcal{N}(P) \subseteq \mathcal{M}_{uo}(A, Q)$$

then we finish the proof

Sylvester operator

given square matrices A and B, a mapping $S : \mathbf{R}^n \to \mathbf{R}^n$

$$S(X) = AX + XB$$

is called a **Sylvestor operator**

Fact: $S(X)$ is singular if A and $-B$ share some common eigenvalues

Proof. suppose λ is a common eigenvalue of A and $-B$

$$Av = \lambda v, \quad w^* B = -\lambda w^*$$

we can construct $X = vw^* \neq 0$ and see that

$$S(X) = Avw^* + vw^* B = \lambda vw^* - \lambda vw^* = 0$$

Uniqueness of stabilizing solution

there is *at most* one solution P of the ARE that yields

$$\mathcal{A} = A - BR^{-1}B^*P \quad \text{stable}$$

Proof suppose there exist two solutions P_1 and P_2 such that

$$\mathcal{A}_1 = A - BR^{-1}B^*P_1 \quad \text{and} \quad \mathcal{A}_2 = A - BR^{-1}B^*P_2 \quad \text{stable}$$

it is easy to verify that

$$(P_1 - P_2)\mathcal{A}_1 + \mathcal{A}_2^*(P_1 - P_2) = 0$$

Recall: the Lyapunov $\mathcal{L}(P) = A^*P + PA$ is singular if A and $-A^*$ share some common eigenvalues

since both $\mathcal{A}_1$ and $\mathcal{A}_2$ are stable, the only solution is $P_1 - P_2 = 0$

Continuous-time infinite horizon LQR problem

Problem: find $u : [0, \infty) \to \mathbf{R}^m$ which minimizes

$$J(x(0), u) = \int_0^\infty x(t)^* Q x(t) + u(t)^* R u(t) dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t)$ given $x(0) \neq 0$

- $Q \succeq 0$ is the state cost matrix
- $R \succ 0$ is the input cost matrix

Boundedness of the cost function

Fact: $J_{\min} < \infty$ implies the existence of a *nonnegative* solution to ARE

any of the following conditions ensures $J_{\min} < \infty$

1. A is stable
2. (A, B) is controllable
3. (A, B) is stabilizable

Proof 1

if A is stable, we would pick $u(\cdot) = 0$ and $x(t) = e^{At}x(0) \to 0$

therefore

$$J_{\min} \leq J(x(0), u(t) = 0) = \int_0^\infty x(t)^* Q x(t) dt < \infty$$

Proof 2 if (A, B) controllable,

- there exists a $u(\cdot)$ such that $u(\cdot)$ steers $x(0)$ to the zero state at time T
- therefore, extend this $u(\cdot)$ such that $u(t) = 0$ for $t > T$
- then of course, $J_{\min} < J(x(0), u) < \infty$
- controllability ensures boundedness of $J_{\min}$ whether A is stable or not

Proof 3 if (A, B) is stabilizable, we have

$$e^{(A+BF)t}x(0) \to 0, \quad t \to \infty$$

for some stabilizing feedback matrix F

therefore

$$J_{\min} < J(x(0), Fx(\cdot)) < \infty$$

(A could be unstable, but the unstable mode must be controllable)

LQR solution

assume P is a **nonnegative** solution to $PA + A^*P - PBR^{-1}B^*P + Q = 0$

if $Q \succ 0$ **or** if (A, Q) observable, then

1. P is a *unique positive* solution
2. the infinite-time LQR problem admits the optimal input

$$u_{\text{opt}}(t) = -R^{-1}B^*Px_{\text{opt}}(t), \quad t \geq 0$$

   where $x_{\text{opt}}(t)$ satisfies

$$\dot{x}_{\text{opt}}(t) = (A - BR^{-1}B^*P)x_{\text{opt}}(t), \quad x_{\text{opt}}(0) = x(0)$$

   and $\mathcal{A} = A - BR^{-1}B^*P$ is *stable*

3. the optimal cost function is

$$J(x(0), u_{\text{opt}}) = x(0)^*Px(0)$$

Solving ARE via Hamiltonian

define $K = -R^{-1}B^*P$

$$\begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^*P \\ -Q - A^*P \end{bmatrix} = \begin{bmatrix} A + BK \\ -Q - A^*P \end{bmatrix}$$

and so

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1}B^* \\ 0 & -(A + BK)^* \end{bmatrix}$$

where 0 in the lower left corner comes from ARE

also note that

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

Hamiltonian matrix is defined by

$$H = \begin{bmatrix} A & -BR^{-1}B^* \\ -Q & -A^* \end{bmatrix}$$

define $\mathcal{A} = A + BK$ and its eigenvalues are $\lambda_1, \ldots, \lambda_n$

- eigenvalues of H are $\lambda_1, \ldots, \lambda_n$ and $-\lambda_1, \ldots, -\lambda_n$
- if T diagonalizes $\mathcal{A}$, *i.e.*, $T^{-1}\mathcal{A}T = \Lambda$, then one can show

$$H \begin{bmatrix} T \\ PT \end{bmatrix} = \begin{bmatrix} T \\ PT \end{bmatrix} \Lambda$$

follow from

$$H \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} A + BK \\ -Q - A^*P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \mathcal{A} = \begin{bmatrix} I \\ P \end{bmatrix} T\Lambda T^{-1}$$

hence, we can compute $2n$ eigenvectors of H, which have the form

$$\mathbf{v}_i = \begin{bmatrix} \mathbf{x}_i \\ \mathbf{y}_i \end{bmatrix}, \quad i = 1, 2, \ldots, 2n$$

collect n eigenvectors associated with n distinct eigenvalues and define

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \ldots & \mathbf{y}_n \end{bmatrix}$$

then every solution of ARE has the form

$$P = YX^{-1}$$

(by selection of subsets of $2n$ eigenvectors of H) provided that X^{-1} exists

Remark: the positive definite P corresponds to **stable** eigenvalues of H

example: let

$$A = \begin{bmatrix} 0 & \sqrt{6} \\ -\sqrt{6} & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{5} \\ 0 \end{bmatrix}$$

- (A, B) is controllable, so there exists a nonnegative solution to ARE
- (A, Q) is observable, so a positive definition solution of ARE is unique
- the eigenvalues of H are $\lambda_1, = 2, \lambda_2 = -2, \lambda_3 = 3, \lambda_4 = -3$
- the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -\sqrt{6}/2 \\ \hline -1 \\ \sqrt{6}/2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ \sqrt{6}/2 \\ \hline 1 \\ \sqrt{6}/2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -\sqrt{6}/3 \\ \hline -1 \\ \sqrt{6}/3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ \sqrt{6}/3 \\ \hline 1 \\ \sqrt{6}/3 \end{bmatrix}$$

case 1: $\lambda_1 = 2, \lambda_2 = -2$

$$P = \begin{bmatrix} 0 & 2/\sqrt{6} \\ \sqrt{6}/2 & 0 \end{bmatrix} \quad \text{(non-self-adjoint)}$$

case 2: $\lambda_1 = 2, \lambda_3 = 3$

$$P = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{(self-adjoint, negative)}$$

case 3: $\lambda_1 = 2, \lambda_4 = -3$

$$P = \begin{bmatrix} 1/5 & 2\sqrt{6}/5 \\ 2\sqrt{6}/5 & -1/5 \end{bmatrix} \quad \text{(self-adjoint, indefinite)}$$

case 4: $\lambda_2 = -2, \lambda_3 = 3$

$$P = \begin{bmatrix} -1/5 & 2\sqrt{6}/5 \\ 2\sqrt{6}/5 & 1/5 \end{bmatrix} \quad \text{(self-adjoint, indefinite)}$$

case 5: $\lambda_2 = -2, \lambda_4 = -3$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(self-adjoint, positive!)}$$

case 6: $\lambda_3 = 3, \lambda_4 = -3$

$$P = \begin{bmatrix} 0 & \sqrt{6}/2 \\ \sqrt{6}/3 & 0 \end{bmatrix} \quad \text{(nonself- adjoint)}$$

- one self-adjoint positive definite solution
- one self-adjoint negative definite solution
- two nonself-adjoint solutions
- two self-adjoint indefinite solutions

the positive definite P is obtained by eigenvectors corresponding to **stable** eigenvalues of the Hamiltonian matrix

Example

design an LQR controller for the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

the system is uncontrollable, but is stabilizable, so $J_{\min} < \infty$

we minimize

$$J = \int_0^\infty x_1^2(t) + u^2(t) dt$$

we have

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1$$

(A, Q) is observable, so there exists a unique positive definite solution P

assume $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, ARE yields

$$1 + 2p_1 - p_1^2 = 0, \quad p_1 - p_1 p_2 = 0, \quad -p_2^2 + 2p_2 - 2p_3 = 0$$

which gives

$$p_1 = 1 \pm \sqrt{2}, \quad p_2 = 1, \quad p_3 = 1/2$$

so, there are two solutions to ARE

$$P_1 = \begin{bmatrix} 1+\sqrt{2} & 1 \\ 1 & 1/2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & 1/2 \end{bmatrix}$$

P_1 is the positive definite solution

if we compute P via the Hamiltonian matrix

there are only 2 combinations of choosing eigenvectors such that X^{-1} exists

Gain margin

consider the effect of varying the gain $K = -R^{-1}B^*P$ on stability

define

$$\mathcal{A}_\sigma = A - \sigma B R^{-1} B^* P$$

when $\sigma > 0$ and P satisfies ARE

Fact: if $Q \succ 0$ or if (A, Q) observable then $\mathcal{A}_\sigma$ is stable for any $\sigma > 1/2$

- LQR provides for one-half gain reduction
- LQR provides infinite gain margin !

Proof. define $\mathcal{A} = A - BR^{-1}B^*P$, so we can write

$$P\mathcal{A}_\sigma = P\mathcal{A} + (1-\sigma)PBR^{-1}B^*P$$

and we have

$$2\text{Re}\langle P\mathcal{A}_\sigma x, x\rangle = 2\text{Re}\langle P\mathcal{A}x, x\rangle + 2(1-\sigma)\|R^{-1/2}B^*Px\|^2$$

by using the ARE, the first term on RHS is

$$2\text{Re}\langle P\mathcal{A}x, x\rangle = -\langle Qx, x\rangle - \|R^{-1/2}B^*Px\|^2$$

hence,

$$\text{Re}\langle P\mathcal{A}_\sigma x, x\rangle = -\langle Qx, x\rangle + (1-2\sigma)\|R^{-1/2}B^*Px\|^2$$

now let x be an eigenvector of $\mathcal{A}_\sigma$, *i.e.*, $\mathcal{A}_\sigma x = \lambda x$, then

$$2\text{Re}\,\lambda\,\langle Px, x\rangle = -\langle Qx, x\rangle + (1-2\sigma)\|R^{-1/2}B^*Px\|^2$$

- since $P \succ 0$ and $\sigma > 1/2$, then Re$\lambda \leq 0$
- if Re $\lambda = 0$, then $Qx = 0$ and $B^* P x = 0$ which implies

$$\mathcal{A}_\sigma x = Ax = \lambda x, \quad \text{and} \;\; Qx = 0, \quad \Longrightarrow (A, Q) \;\; \text{unobservable}$$

so Re $\lambda = 0$ never happens if $Q \succ 0$ or (A, Q) observable !

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