6. Spectral analysis

- power spectral density
- periodogram analysis
- window functions

Power Spectral density

Wiener-Khinchin theorem:

if a process is wide-sense stationary, the autocorrelation function and the power spectral density form a Fourier transform pair:

Continuous

$$S(\omega) = \int_{-\infty}^{\infty} e^{-\mathrm{i}\omega\tau} R(\tau) d\tau \quad \Longleftrightarrow \quad R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{\mathrm{i}\omega t} d\omega$$

Discrete

$$S(\omega) = \sum_{k=-\infty}^{k=\infty} R(k) e^{-\mathrm{i}\omega k} \quad \Longleftrightarrow \quad R(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{\mathrm{i}\omega k} d\omega$$

(under a condition for the existence of the Fourier transform, e.g., $R(t)$ is absolutely integrable or $R(k)$ is absolutely summable)

Properties of PSD

- $S(\omega)$ is self-adjoint, i.e., $S(\omega) = S^*(\omega), \forall \omega$
- $S(\omega) \succeq 0$ for all ω
- $\int_{-\infty}^{\infty} S(\omega) d\omega = R(0) = \mathbf{E}\, x(t)x(t)^* \succeq 0$ (average power)
- for real processes, $S(-\omega) = S(\omega)^T$
- for discrete-time processes, $S(\omega)$ is a periodic function of period 2π

Cross-power spectral density

the cross-power spectrum of $x(t)$ and $y(t)$ is the Fourier transform of the cross correlation $R_{xy}(\tau)$:

Continuous

$$S_{xy}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{xy}(\tau) d\tau \quad \Longleftrightarrow \quad R_{xy}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xy}(\omega) e^{i\omega t} d\omega$$

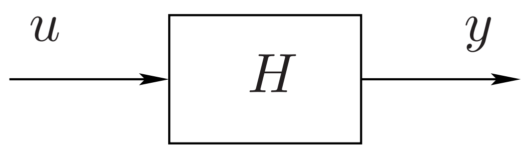
Discrete

$$S_{xy}(\omega) = \sum_{k=-\infty}^{k=\infty} R_{xy}(k) e^{-i\omega k} \quad \Longleftrightarrow \quad R_{xy}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xy}(\omega) e^{i\omega k} d\omega$$

It follows from $R_{xy}(-\tau) = R_{yx}^*(\tau)$ that

$$S_{xy}(\omega) = S_{yx}^*(\omega)$$

LTI systems with random inputs



Fact: if $u(t)$ is wide-sense stationary, $y(t)$ is also wide-sense stationary

- the mean is constant for all t

$$\mathbf{E}\, y(t) = \sum_{s=-\infty}^{\infty} h(s)\, \mathbf{E}\, u(t-s) = \mu_u \sum_{s=-\infty}^{\infty} h(s)$$

- $R_y(t_1, t_2)$ depends only on the time shift $t_1 - t_2$

$$\begin{aligned} R_y(t_1, t_2) &= \sum_{s=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(s)\, \mathbf{E}[u(t_1 - s)u(t_2 - v)^*]h^*(v) \\ &= \sum_{s=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(s) R_u(t_1 - t_2 + v - s) h^*(v) \end{aligned}$$

Fact: $y(t), u(t)$ are jointly wide-sense stationary

the input-output cross correlation is

$$\begin{aligned} R_{yu}(t_1, t_2) &= \mathbf{E} \sum_{k=-\infty}^{\infty} h(k) u(t_1 - k) u(t_2)^* \\ &= \sum_{k=-\infty}^{\infty} h(k) R_u(t_1 - t_2 - k) \\ R_{yu}(\tau) &= \sum_{k=-\infty}^{\infty} h(k) R_u(\tau - k) \end{aligned}$$

it also follows that

$$R_y(\tau) = \sum_{k=-\infty}^{\infty} h(k) R_{uy}(\tau - k)$$

conclusion: the correlations are in the form of convolution sum

Spectral relations for LTI systems

Using the convolution property of the Fourier transform of $R_{yu}(\tau), R_y(\tau)$, we have the relations:

$$S_{yu}(\omega) = H(\omega)S_u(\omega), \quad S_y(\omega) = H(\omega)S_{uy}(\omega)$$

With $S_{uy}(\omega) = S_{yu}^*(\omega)$, we have

$$S_y(\omega) = H(\omega)S_u(\omega)H(\omega)^*$$

In terms of z-transform, this could be written as

$$S_y(z) = H(z)S_u(z)H(z)^*$$

where $H(z)^* = H(\bar{z})^T$ and we should be aware that $z = e^{\mathrm{i}\omega}$ in the analysis

Example 1

suppose the covariance function of a staionary process is given by

$$R(k) = \frac{\lambda^2 a^{|k|}}{1-|a|^2}, \quad |a| < 1, \quad \lambda \in \mathbf{R}$$

the spectral density can be obtained via z-transform

$$\begin{aligned} S(z) &= \frac{\lambda^2}{1-|a|^2} \sum_{k=-\infty}^{\infty} a^{|k|} z^{-k} = \frac{\lambda^2}{1-|a|^2} \left(\sum_{k=-\infty}^{-1} a^{-k} z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k} \right) \\ &= \frac{\lambda^2}{1-|a|^2} \left(\frac{az}{1-az} + \frac{z}{z-a} \right) = \frac{\lambda^2}{(1-az)(1-az^{-1})} \end{aligned}$$

substituting $z = e^{\mathrm{i}\omega}$ gives

$$S(\omega) = \frac{\lambda^2}{(1-ae^{\mathrm{i}\omega})(1-ae^{-\mathrm{i}\omega})} = \frac{\lambda^2}{1+a^2-2a\cos\omega}$$

Example 2

a linear system given in a state-space form

$$y(t) = ay(t-1) + e(t)$$

where $e(t)$ is a white noise with variance λ^2

the transfer function is given by

$$H(z) = \frac{1}{1 - az^{-1}}$$

therefore the spectral density of y is

$$S_y(\omega) = \frac{\lambda^2}{(1 - ae^{-\mathrm{i}\omega})(1 - ae^{\mathrm{i}\omega})} = \frac{\lambda^2}{1 + a^2 - 2a\cos\omega}$$

Spectral analysis

use the same model as in correlation analysis:

$$R_{yu}(\tau) = \sum_{k=0}^{\infty} h(k) R_u(\tau - k)$$

taking DFT gives the spectral representation

$$S_{yu}(\omega) = H(\omega) S_u(\omega)$$

if $S_u(\omega) \succ 0$ for all ω, then we can estimate

$$\hat{H}(\omega) = \hat{S}_{yu}(\omega) \hat{S}_u(\omega)^{-1},$$

where $\hat{S}_{yu}, \hat{S}_u$ can be computed via DFT

Periodogram analysis

an infinite-length discrete-time signal $y(t)$ is windowed by a length-N window $w(t)$, $1 \le t \le N$

$$\tilde{y}(t) = w(t)y(t)$$

define a function $Y_N(\omega)$ given by

$$Y_N(\omega) = \frac{1}{\sqrt{N}} \sum_{t=1}^{N} w(t)y(t)e^{-\mathrm{i}\omega t}$$

the *periodogram*, an estimate of $S_y(\omega)$, is defined by

$$\hat{S}_y(\omega) = \frac{1}{C}|Y_N(\omega)|^2,$$

where $C = \frac{1}{N}\sum_{t=1}^{N} |w(t)|^2$ is a normalization factor

Periodogram analysis

$\hat{S}_y(\omega)$ is called *periodogram* when $w(t)$ is rectangular, and *modified periodogram* for other types of windows, e.g., Hamming, Barlett, etc.

in practice, the periodogram is evaluated at a finite number of frequencies

$$\omega_k = 2\pi k/R, \quad 0 \leq k \leq R-1$$

by replacing $\hat{S}_y(\omega)$ with the length-R DFT $Y[k]$ of the length-N sequences $y[k]$:

$$\hat{S}_y(\omega_k) = \hat{S}_y[k] = \frac{1}{C}|Y[k]|^2$$

- usually $R > N$ to provide a finer resolution of the periodogram
- $C = (1/N)\sum_{t=1}^{N} |w(t)|^2$ is a normalization factor

Window functions

suppose we use a rectangular window of length N

$$
\begin{aligned}
\hat{S}_y(\omega) &= \frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N} y(m)y^*(n)e^{-\mathrm{i}\omega(m-n)} \\
&= \frac{1}{N}\sum_{k=-N+1}^{N-1}\sum_{n=1-k}^{N-k} y(n+k)y^*(n)e^{-\mathrm{i}\omega k} \\
&= \sum_{k=-N+1}^{N-1} \hat{R}_y(k)e^{-\mathrm{i}\omega k}
\end{aligned}
$$

- the periodogram is the Fourier transform of $\hat{R}_y(k)$
- a few samples of $y(n)$ is used in estimating $\hat{R}_y(k)$ when k is large, yielding a poor estimate of $R_y(k)$

Window functions

use the window functions that vanish for $|\tau| > M$ to weight out the estimated correlation for large τ

- Rectangular

$$w(\tau) = 1, \quad |\tau| \leq M$$

- Barlett

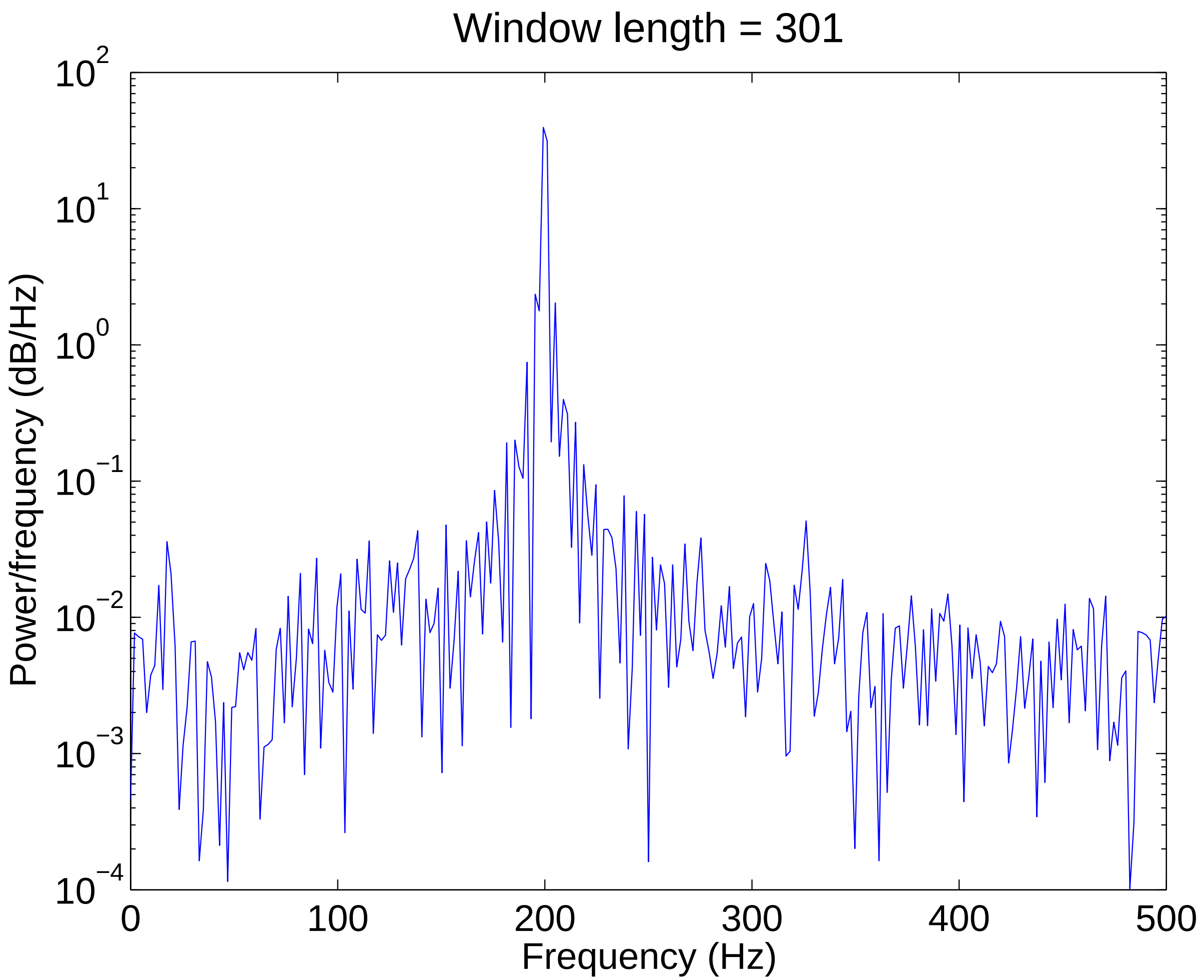
$$w(\tau) = 1 - |\tau|/M, \quad |\tau| \leq M$$

- Hamming

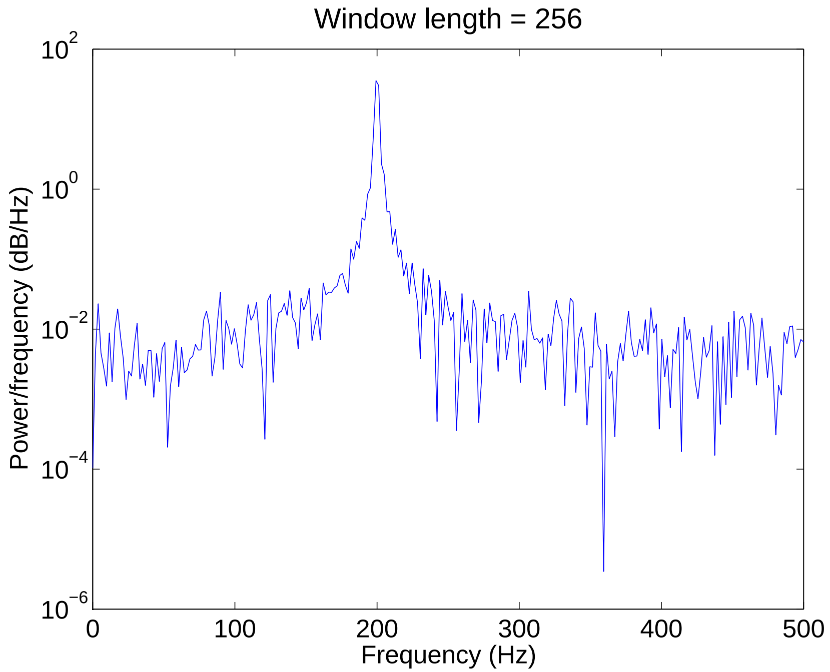
$$w(\tau) = 0.54 + 0.46 \cos\left(\frac{2\pi\tau}{2M+1}\right), \quad |\tau| \leq M$$

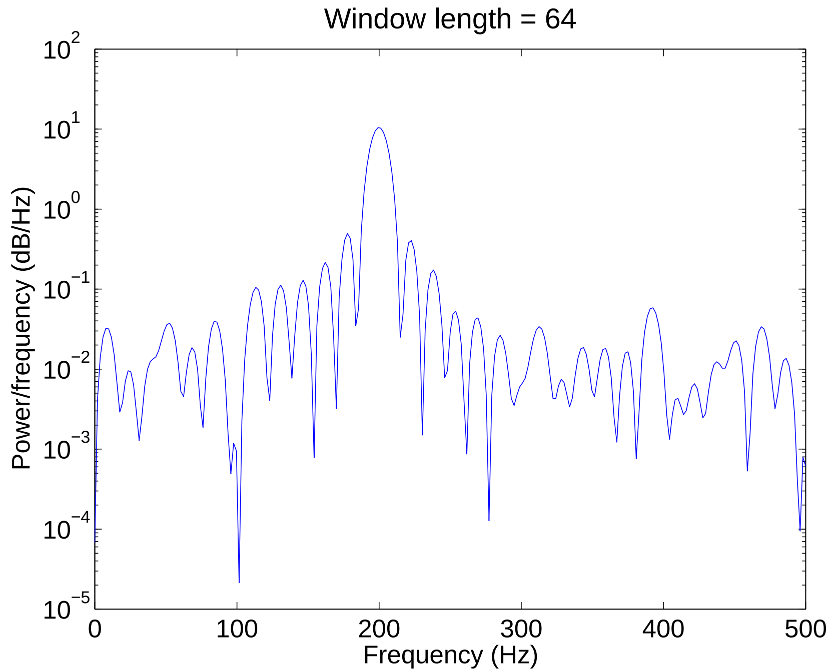
M should be small compared to N to reduce the fluctuations of the periodogram

Example



- $y(t) = \cos(400\pi t) + \nu(t)$, with $N = 301$


Window length = 256
Power/frequency (dB/Hz)
Frequency (Hz)
10^2
10^0
10^−2
10^−4
10^−6
0
100
200
300
400
500


Window length = 64
Power/frequency (dB/Hz)
Frequency (Hz)
10^2
10^1
10^0
10^−1
10^−2
10^−3
10^−4
10^−5
0
100
200
300
400
500

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