

7. Input signals

- Common input signals in system identification
 - step function
 - sum of sinusoids
 - ARMA sequences
 - pseudo random binary sequence (PRBS)
- spectral characteristics
- persistent excitation

Step function

a step function is given by

$$u(t) = \begin{cases} 0, & t < 0 \\ u_0, & t \geq 0 \end{cases}$$

where the amplitude u_0 is arbitrarily chosen

- related to rise time, overshoots, static gain, etc.
- useful for systems with a large signal-to-noise ratio

Sum of sinusoids

the input signal $u(t)$ is given by

$$u(t) = \sum_{k=1}^m a_k \sin(\omega_k t + \phi_k)$$

where the angular frequencies $\{\omega_k\}$ are distinct,

$$0 \leq \omega_1 < \omega_2 < \dots < \omega_m \leq \pi$$

and the amplitudes and phases a_k, ϕ_k are chosen by the user

Characterization of sinusoids

let S_N be the average of a sinusoid over N points

$$S_N = \frac{1}{N} \sum_{t=1}^N a \sin(\omega t + \phi)$$

Let μ be the mean of the sinusoidal function

$$\mu = \lim_{N \rightarrow \infty} S_N = \begin{cases} a \sin \phi, & \omega = 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

- $u(t) = \sum_{k=1}^m a_k \sin(\omega_k t + \phi_k)$ has zero mean if $\omega_1 > 0$
- WLOG, assume zero mean for $u(t)$ (we can always subtract the mean)

Spectrum of sinusoidal inputs

the autocorrelation function can be computed by

$$\begin{aligned} R(\tau) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N u(t + \tau)u(t) \\ &= \sum_{k=1}^m C_k \cos(\omega_k \tau) \end{aligned}$$

with $C_k = a_k^2/2$ for $k = 1, 2, \dots, m$

if $\omega_m = \pi$, the coefficient C_m should be modified by

$$C_m = a_m^2 \sin^2 \phi_m$$

therefore, the spectrum is

$$S(\omega) = \sum_{k=1}^m (C_k/2) [\delta(\omega - \omega_k) + \delta(\omega + \omega_k)]$$

Autoregressive Moving Average sequence

let $e(t)$ be a pseudorandom sequence similar to white noise in the sense that

$$\frac{1}{N} \sum_{t=1}^N e(t)e(t + \tau) \rightarrow 0, \quad \text{as } N \rightarrow \infty$$

a general input $u(t)$ can be obtained by linear filtering

$$u(t) + c_1u(t - 1) + \dots + c_pu(t - p) = e(t) + d_1e(t - 1) + \dots + d_qe(t - p)$$

- $u(t)$ is called *ARMA (autoregressive moving average)* process
- when all $c_i = 0$ it is called *MA (moving average)* process
- when all $d_i = 0$ it is called *AR (autoregressive)* process
- the user gets to choose c_i, d_i and the random generator of $e(t)$

the transfer function from $e(t)$ to $u(t)$ is

$$U(z) = \frac{D(z)}{C(z)}E(z)$$

where

$$C(z) = 1 + c_1z^{-1} + c_2z^{-2} + \dots c_pz^{-p}$$
$$D(z) = 1 + d_1z^{-1} + d_2z^{-2} + \dots d_qz^{-q}$$

- the distribution of $e(t)$ is often chosen to be Gaussian
- c_i, d_i are chosen such that $C(z), D(z)$ have zeros outside the unit circle
- different choices of c_i, d_i lead to inputs with various spectral characteristics

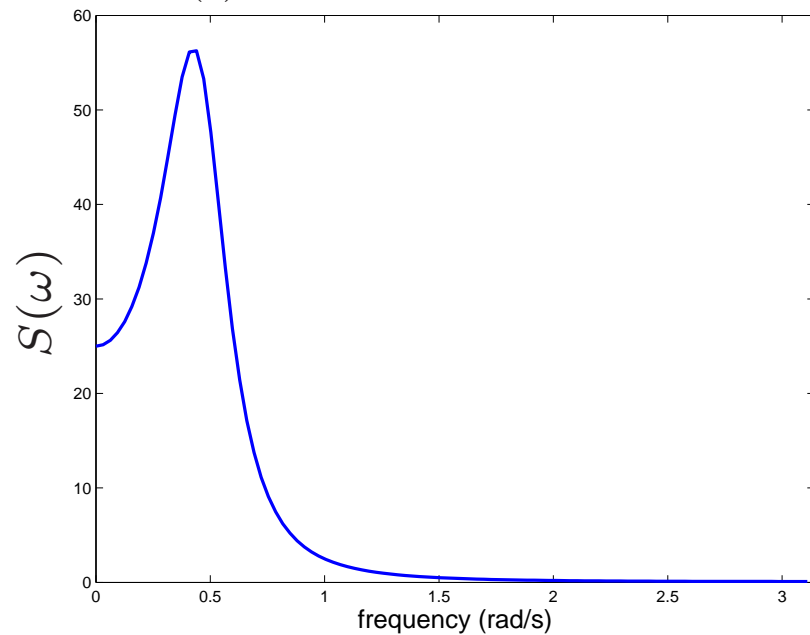
Spectrum of ARMA process

let $e(t)$ be a white noise with variance λ^2

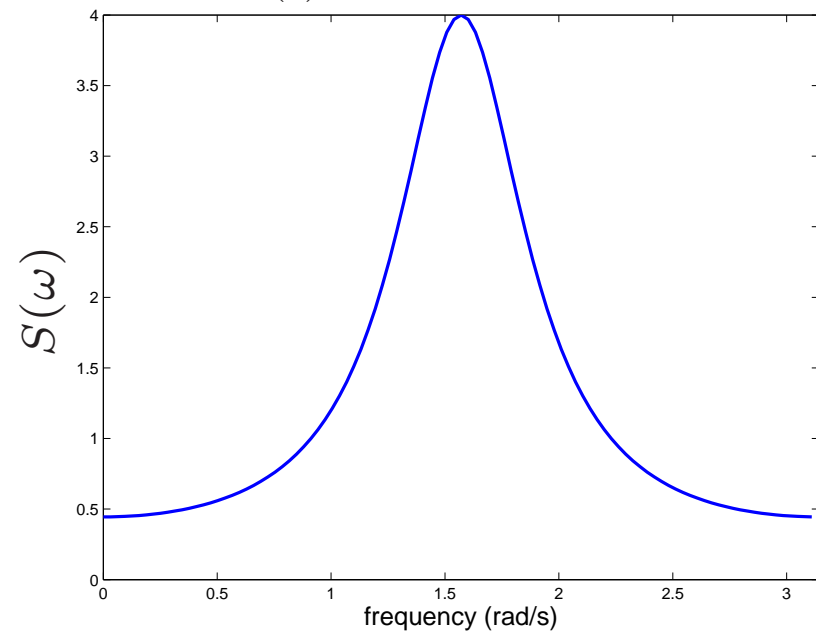
the spectral density of ARMA process is

$$S(\omega) = \lambda^2 \left| \frac{D(\omega)}{C(\omega)} \right|^2$$

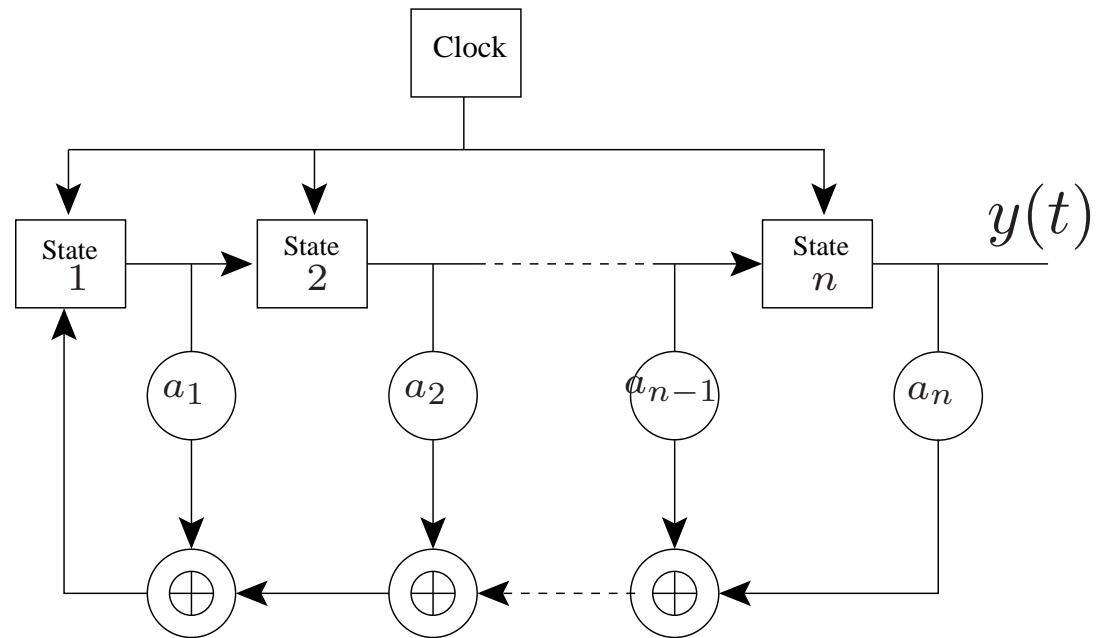
$$\frac{D(z)}{C(z)} = 1/(z^2 - 1.5z + 0.7)$$



$$\frac{D(z)}{C(z)} = 1/(z^2 + 0.5)$$



Pseudorandom Binary Sequence (PRBS)



$$x(t+1) = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix} x(t)$$

$$y(t) = [0 \quad \dots \quad 0 \quad 1] x(t)$$

Characteristics of PRBS

- every initial state is allowed except the all-zero state
- the feedback coefficients a_1, a_2, \dots, a_n are either 0 or 1
- all additions is modulo-two operation
- the sequences are two-state signals (binary)
- there are possible $2^n - 1$ different state vectors x (all-zero state is excluded)
- a PRBS of period equal to $M = 2^n - 1$ is called a **maximum length PRBS** (ML PRBS)
- for *maximum length PRBS*, its characteristic resembles white random noise (pseudorandom)

Influence of the Feedback Path

let $n = 3$ and initialize x with $x(0) = (1, 0, 0)$

- with $a = (1, 1, 0)$, the state vectors $x(k), k = 1, 2, \dots$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

the sequence has period equal to 3

- with $a = (1, 0, 1)$, the state vectors $x(k), k = 1, 2, \dots$ are

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the sequence has period equal to 7 (the maximum period, $2^3 - 1$)

Maximum length PRBS

denote q^{-1} the unit delay operator and let

$$A(q^{-1}) = 1 \oplus a_1q^{-1} \oplus a_2q^{-2} \oplus \dots \oplus a_nq^{-n}$$

the PRBS $y(t)$ satisfies the homogeneous equation:

$$A(q^{-1})y(t) = 0$$

this equation has only solutions of period $M = 2^n - 1$ *if and only if*

1. the binary polynomial $A(q^{-1})$ is irreducible, *i.e.*, there do not exist any two polynomial $A_1(q^{-1})$ and $A_2(q^{-1})$ such that

$$A(q^{-1}) = A_1(q^{-1})A_2(q^{-1})$$

2. $A(q^{-1})$ is a factor of $1 \oplus q^{-M}$ but is not a factor of $1 \oplus q^{-p}$ for any $p < M$

Generating Maximum length PRBS

examples of polynomials $A(z)$ satisfying the previous two conditions

n	$A(z)$
3	$1 \oplus z \oplus z^3$
4	$1 \oplus z \oplus z^4$
5	$1 \oplus z^2 \oplus z^5$
6	$1 \oplus z \oplus z^6$
7	$1 \oplus z \oplus z^7$
8	$1 \oplus z \oplus z^2 \oplus z^7 \oplus z^8$
9	$1 \oplus z^4 \oplus z^9$
10	$1 \oplus z^3 \oplus z^{10}$

Properties of maximum length PRBS

Let $y(t)$ be an ML PRBS of period $M = 2^n - 1$

- within one period $y(t)$ contains $(M + 1)/2 = 2^{n-1}$ ones and $(M - 1)/2 = 2^{n-1} - 1$ zeros
- For $k = 1, 2, \dots, M - 1$,

$$y(t) \oplus y(t - k) = y(t - l)$$

for some $l \in [1, M - 1]$ that depends on k

moreover, for any binary variables x, y ,

$$xy = \frac{1}{2} (x + y - (x \oplus y))$$

these properties will be used to compute the covariance function of maximum length PRBS

Covariance function of maximum length PRBS

the mean is given by counting the number of outcome 1 in $y(t)$:

$$m = \frac{1}{M} \sum_{t=1}^M y(t) = \frac{1}{M} \left(\frac{M+1}{2} \right) = \frac{1}{2} + \frac{1}{2M}$$

the mean is slightly greater than 0.5

using $y^2(t) = y(t)$, we have the covariance function at lag zero as

$$C(0) = \frac{1}{M} \sum_{t=1}^M y^2(t) - m^2 = m - m^2 = \frac{M^2 - 1}{4M^2}$$

the variance is therefore slightly less than $1/4$

Covariance function of maximum length PRBS

For $\tau = 1, 2, \dots,$

$$\begin{aligned} C(\tau) &= (1/M) \sum_{t=1}^M y(t + \tau)y(t) - m^2 \\ &= \frac{1}{2M} \sum_{t=1}^M [y(t + \tau) + y(t) - (y(t + \tau) \oplus y(t))] - m^2 \\ &= m - \frac{1}{2M} \sum_{t=1}^M y(t + \tau - 1) - m^2 = m/2 - m^2 \\ &= -\frac{M + 1}{4M^2} \end{aligned}$$

Asymptotic behavior of the covariance function of PRBS

Define $\tilde{y}(t) = -1 + 2y(t)$ so that its outcome is either -1 or 1

if M is large enough,

$$\tilde{m} = -1 + 2m = 1/M \approx 0$$

$$\tilde{C}(0) = 4C(0) = 1 - 1/M^2 \approx 1$$

$$\tilde{C}(\tau) = 4C(\tau) = -1/M - 1/M^2 \approx -1/M, \quad \tau = 1, 2, \dots, M - 1$$

with a large period length M

- the covariance function of PRBS has similar properties to a white noise
- however, their spectral density matrices can be drastically different

Spectral density of PRBS

the output of PRBS sequence is shifted to values $-a$ and a with period M
the autocorrelation function is also periodic and given by

$$R(\tau) = \begin{cases} a^2, & \tau = 0, \pm M, \pm 2M, \dots \\ -\frac{a^2}{M}, & \text{otherwise} \end{cases}$$

since $R(\tau)$ is periodic with period M , it has a Fourier representation:

$$R(\tau) = \sum_{k=0}^{M-1} C_k e^{i2\pi\tau k/M}, \quad \text{with Fourier coefficients } C_k$$

therefore, the spectrum of PRBS is an impulse train:

$$S(\omega) = \sum_{k=0}^{M-1} C_k \delta\left(\omega - \frac{2\pi k}{M}\right)$$

Spectral density of PRBS

hence, the Fourier coefficients

$$C_k = \frac{1}{M} \sum_{\tau=0}^{M-1} R(\tau) e^{-i2\pi\tau k/M}$$

are also the spectral coefficients of $S(\omega)$

using the expression of $R(\tau)$, we have

$$C_0 = \frac{a^2}{M^2}, \quad C_k = \frac{a^2}{M^2}(M+1), \quad k = 1, 2, \dots$$

therefore,

$$S(\omega) = \frac{a^2}{M^2} \left[\delta(\omega) + (M+1) \sum_{k=1}^{M-1} \delta(\omega - 2\pi k/M) \right]$$

It does not resemble spectral characteristic of a white noise (flat spectrum)

Comparison of the covariances between filtered inputs

- define $y_1(t)$ as the output of a filter:

$$y_1(t) - ay_1(t - 1) = u_1(t),$$

with white noise $u(t)$ of zero mean and variance λ^2

- define $y_2(t)$ be the output of the same filter:

$$y_2(t) - ay_2(t - 1) = u_2(t),$$

where $u_2(t)$ is a PRBS of period M and amplitude λ

what can we say about the covariances of $y_1(t)$ and $y_2(t)$?

Comparison of the correlations between filtered inputs

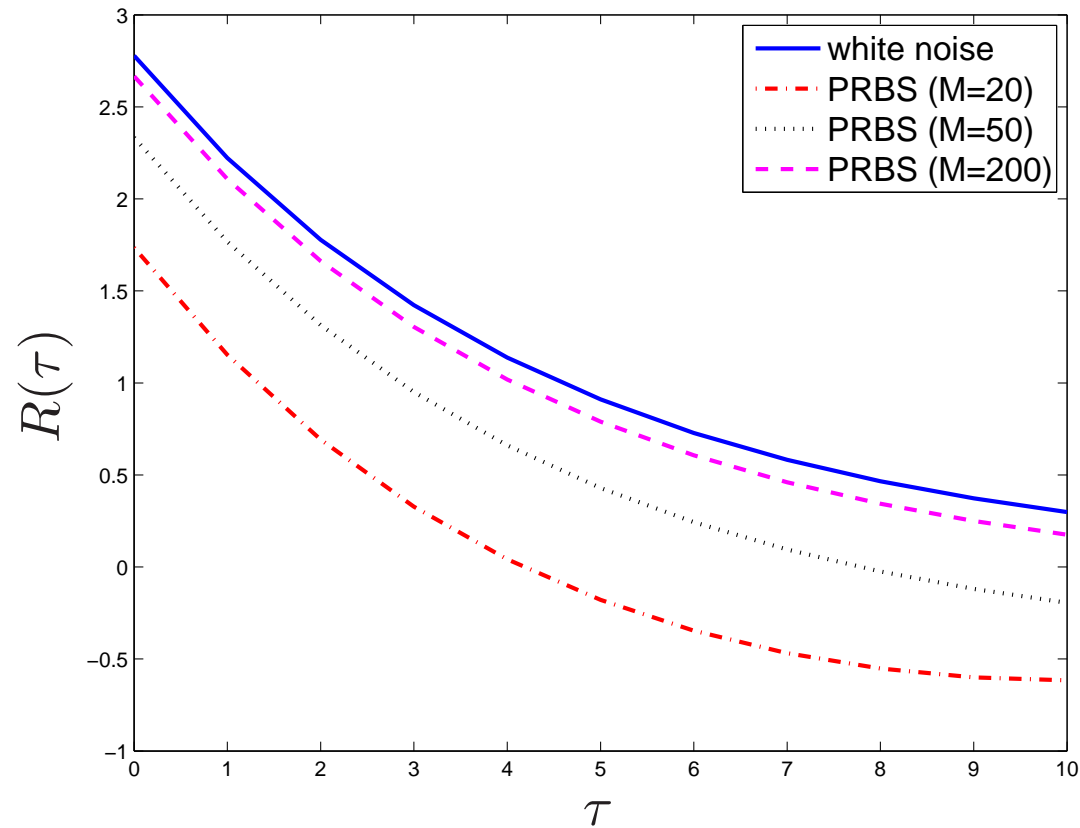
the correlation function of $y_1(t)$ is given by

$$R_1(\tau) = \frac{\lambda^2}{1 - a^2} a^\tau, \quad \tau \geq 0$$

the correlation function of $y_2(t)$ can be calculated as

$$\begin{aligned} R_2(\tau) &= \int_{-\pi}^{\pi} S_{y_2}(\omega) e^{i\omega\tau} d\omega \\ &= \int_{-\pi}^{\pi} S_{u_2}(\omega) \left| \frac{1}{1 - ae^{i\omega}} \right|^2 e^{i\tau\omega} d\omega \\ &= \frac{\lambda^2}{M} \left[\frac{1}{(1 - a)^2} + (M + 1) \sum_{k=1}^{M-1} \frac{\cos(2\pi\tau k/M)}{1 + a^2 - 2a \cos(2\pi k/M)} \right] \end{aligned}$$

Plots of the correlation functions



- the filter parameter is $a = 0.8$
- $R(\tau)$ of white noise and PRBS inputs are very close when M is large

Persistent excitation

a signal $u(t)$ is *persistently exciting* of order n if

1. the following limit exists:

$$R(\tau) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N u(t + \tau) u(t)^*$$

2. the following matrix is positive definite

$$\mathbf{R}_n = \begin{bmatrix} R(0) & R(1) & \dots & R(n-1) \\ R(-1) & R(0) & \dots & R(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ R(1-n) & R(2-n) & \dots & R(0) \end{bmatrix}$$

(if $u(t)$ is from an ergodic stochastic process, then $\mathbf{R}(n)$ is the usual covariance matrix (assume zero mean))

Examining the order of persistent excitation

- **white noise input** of zero mean and variance λ^2

$$R(\tau) = \lambda^2 \delta(\tau), \quad \implies \quad \mathbf{R}_n = \lambda^2 \mathbf{I}_n$$

thus, white noise is persistently exciting of *all* orders

- **step input** of magnitude λ

$$R(\tau) = \lambda^2, \quad \forall \tau \quad \implies \quad \mathbf{R}_n = \lambda^2 \mathbf{1}_n$$

a step function is persistently exciting of order 1

- **impulse input:** $u(t) = 1$ for $t = 0$ and 0 otherwise

$$R(\tau) = 0, \quad \forall \tau \quad \implies \quad \mathbf{R}_n = 0$$

an impulse is *not* persistently exciting of any order

Example 1: FIR models

recall the problem of estimating an FIR model where

$$h(k) = 0, \quad k \geq M$$

the coefficients $h(k)$ are the solution to the following equation

$$\begin{bmatrix} R_{yu}^*(0) \\ R_{yu}^*(1) \\ \vdots \\ R_{yu}^*(M-1) \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(1) & \cdots & R_u(M-1) \\ R_u(-1) & R_u(0) & \cdots & R_u(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ R_u(1-M) & R_u(2-M) & \cdots & R_u(0) \end{bmatrix} \begin{bmatrix} h^*(0) \\ h^*(1) \\ \vdots \\ h^*(M-1) \end{bmatrix}$$

- the equations has a unique solution iff \mathbf{R}_M is nonsingular
- equivalent condition: u must be persistently exciting of order M

Example 2: Estimating noisy linear models

consider a least-squares problem of estimating a first-order model

$$y(t) = ay(t - 1) + bu(t) + e(t)$$

where $u(t)$ is an input signal, and $e(t)$ is an i.i.d. noise of zero mean

we can show that

- if $u(t)$ is a PRBS or step input, the consistent estimates are obtained, i.e.,

$$(\hat{a}, \hat{b}) \rightarrow (a, b), \quad \text{as } N \rightarrow \infty$$

- if $u(t)$ is an impulse, $\hat{a} \rightarrow a$ but \hat{b} does not converge to b as N increases
- in loose terms, the impulse input does not provide enough information on $y(t)$ to estimate b

Properties of persistently exciting signals

assumptions:

- $u(t)$ is a multivariable ergodic process
- $S_u(\omega)$ is positive in at least n distinct frequencies within $(-\pi, \pi)$

we have the following two properties

Property 1 $u(t)$ is persistently exciting of order n

Property 2 the filtered signal $y(t) = H(q^{-1})u(t)$ is persistently exciting of order n

where $H(z)$ is an asymptotically stable linear filter and $\det H(z)$ has no zero on the unit circle

from above facts, we can imply

an ARMA process is persistently exciting of any finite order

Examining the order of PRBS

consider a PRBS of period M and magnitude $a, -a$

the matrix containing n -covariance sequences (where $n \leq M$) is

$$\mathbf{R}_n = \begin{bmatrix} a^2 & -a^2/M & \dots & -a^2/M \\ -a^2/M & a^2 & \dots & -a^2/M \\ \vdots & \vdots & \ddots & \vdots \\ -a^2/M & -a^2/M & \dots & a^2 \end{bmatrix}$$

for any $x \in \mathbf{R}^n$,

$$\begin{aligned} x^T \mathbf{R}_n x &= x^T \left(\left(a^2 + \frac{a^2}{M} \right) I - \frac{a^2}{M} \mathbf{1} \mathbf{1}^T \right) x \\ &\geq a^2 \left(1 + \frac{1}{M} \right) x^T x - \frac{a^2}{M} x^T x \mathbf{1}^T \mathbf{1} = a^2 \|x\|^2 \left(1 + \frac{(1-n)}{M} \right) \geq 0 \end{aligned}$$

a PRBS with period M is persistently exciting of order M

Examining the order of sum of sinusoids

consider the signal $u(t) = \sum_{k=1}^m a_k \sin(\omega_k t + \phi_k)$

where $0 \leq \omega_1 < \omega_2 < \dots < \omega_m \leq \pi$

the spectral density of u is given by

$$S(\omega) = \sum_{k=1}^m \frac{C_k}{2} [\delta(\omega - \omega_k) + \delta(\omega + \omega_k)]$$

therefore $S(\omega)$ is nonzero (in the interval $(-\pi, \pi]$) in exactly n points where

$$n = \begin{cases} 2m, & 0 < \omega_1, \omega_m < \pi \\ 2m - 1, & 0 = \omega_1, \text{ or } \omega_m = \pi \\ 2m - 2, & 0 = \omega_1 \text{ and } \omega_m = \pi \end{cases}$$

it follows from Property 1 that $u(t)$ is persistently exciting of order n

Summary

- the choice of input is imposed by the type of identification method
- the input signal should be persistently exciting of a certain order to ensure that the system can be identified
- some often used signals include PRBS and ARMA processes

References

Chapter 5 in

T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989