4. Correlation Analysis

- analysis on LTI systems
- finite impulse response (FIR) model

Correlation analysis

consider a discrete LTI system with a disturbance $v(t)$

$$y(t) = \sum_{k=0}^{\infty} h(k)u(t-k) + v(t)$$

assume u, v have zero mean and $\mathbf{E}\, u(t)v(s)^* = 0, \forall t, s$.

the correlation function is given by

$$R_{yu}(\tau) = \mathbf{E}\, y(t+\tau)u(t)^* = \sum_{k=0}^{\infty} h(k)R_u(\tau - k)$$

If $u(t)$ is *white noise* ($R_u(\tau) = 0, \tau \neq 0$), it is simplified to

$$R_{yu}(k) = h(k)R_u(0)$$

use finite approximation of $R_{yu}(k)$ and $R_u(0)$ to solve for $h(k)$

$$\hat{R}_{yu}(\tau) = \frac{1}{N}\sum_{t=1}^{N-\tau} y(t+\tau)u(t)^*, \quad \tau = 0, 1, 2, \ldots$$

$$\hat{R}_{uu}(\tau) = \frac{1}{N}\sum_{t=1}^{N-\tau} u(t+\tau)u(t)^*, \quad \tau = 0, 1, 2, \ldots$$

when $u(t)$ is not exactly white

- filter both inputs and outputs that makes the input as white as possible
- truncate the impulse response at a certain order

Finite Impulse Response (FIR) Models

assume that

$$h(k) = 0, \quad k > M$$

this is called *a finite impulse respose (FIR)* or a *truncated weighting function*

the correlation equation becomes

$$R_{yu}(\tau) = \sum_{k=0}^{M} h(k) R_u(\tau - k)$$

Writing out this equation for $\tau = 0, 1, \ldots, M$ gives a linear equation:

$$\begin{bmatrix} R^*_{yu}(0) \\ R^*_{yu}(1) \\ \vdots \\ R^*_{yu}(M) \end{bmatrix} = \begin{bmatrix} R_u(0) & R_u(1) & \cdots & R_u(M) \\ R_u(-1) & R_u(0) & \cdots & R_u(M-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_u(-M) & R_u(-M+1) & \cdots & R_u(0) \end{bmatrix} \begin{bmatrix} h^*(0) \\ h^*(1) \\ \vdots \\ h^*(M) \end{bmatrix}$$

Example with white noise input

consider a scalar system

$$x(t) + ax(t-1) = bu(t-1), \quad |a| < 1$$
$$y(t) = x(t) + v(t)$$

with $a = 0.5, b = 5$

assume that $u(t)$ and $v(t)$ are independent white noise with variances $\sigma_u^2 = \sigma_v^2 = 0.1$

The transfer function is

$$H(z) = \frac{bz^{-1}}{1 + az^{-1}} = b(z^{-1} - az^{-2} + a^2 z^{-3} - a^3 z^{-4} + \ldots)$$

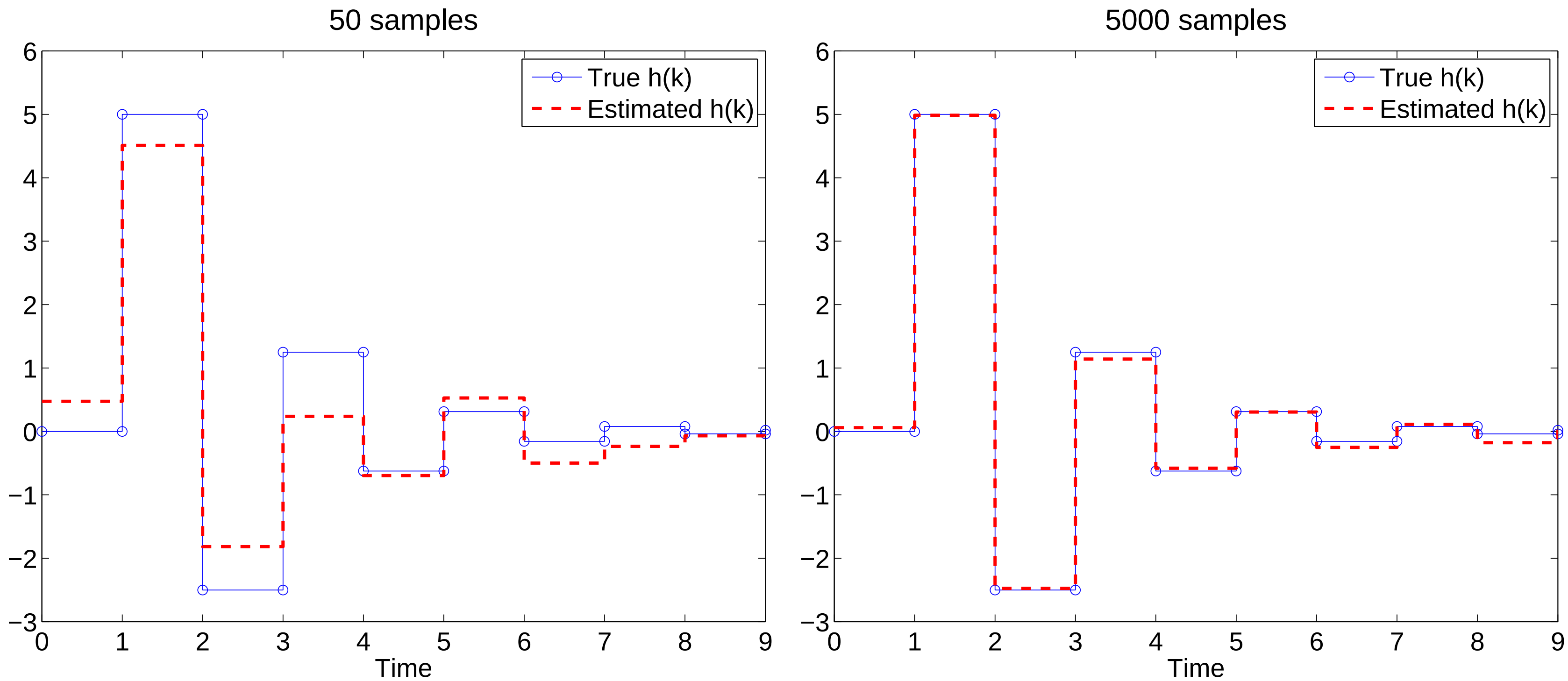
The impulse response is therefore given by

$$h(0) = 0, \quad h(k) = b(-a)^{k-1}, k \geq 1$$

Example with white noise input

the estimate of the impulse response is

$$\hat{h}(k) = \hat{R}_{yu}(k)/\hat{\sigma}_u^2$$



$\hat{\sigma}_u^2 = 0.093$ $\hat{\sigma}_u^2 = 0.099$

References

Chapter 6 in
L. Ljung, *System Identification: Theory for the User*, Prentice Hall, Second edition, 1999

Chapter 3 in
T. Söderström and P. Stoica, *System Identification*, Prentice Hall, 1989