Linear algebra and applications

Jitkomut Songsiri

Department of Electrical Engineering
Faculty of Engineering
Chulalongkorn University

CUEE

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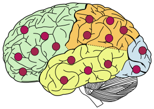
Outline

How to read this handout

1. the note is used with lecture in EE205 (you cannot master this topic just by reading this note) – class activities include
   - graphical concepts, math derivation of details/steps in between
   - computer codes to illustrate examples
2. always read 'textbooks' after lecture
3. pay attention to the symbol ✎; you should be able to prove such ✎ result
4. each chapter has a list of references; find more formal details/proofs from in-text citations
5. almost all results in this note can be Googled; readers are encouraged to 'stimulate neurons' in your brain by proving results without seeking help from the Internet first
6. typos and mistakes can be reported to `jitkomut@gmail.com`



System of linear equations

System of linear equations

a linear system of m equations in n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

in matrix form: $Ax = b$

problem statement: given A, b, find a solution x (if exists)

Example: solving ordinary differential equations

given $y(0) = 1, \dot{y}(0) = -1, \ddot{y}(0) = 0$, solve

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 0$$

the closed-form solution is

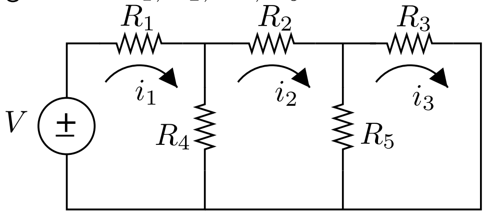
$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{-3t}$$

C_1, C_2 and C_3 can be found by solving a set of linear equations

$$\begin{aligned} 1 &= y(0) &&= C_1 + C_2 + C_3 \\ -1 &= \dot{y}(0) &&= -C_1 - 2C_2 - 3C_3 \\ 0 &= \ddot{y}(0) &&= C_1 + 4C_2 + 9C_3 \end{aligned}$$

Example: linear static circuit

given V, $R_1, R_2, \ldots, R_5$, find the currents in each loop



$$
\begin{aligned}
V &= (R_1 + R_4)i_1 - R_4 i_2 \\
0 &= -R_4 i_1 + (R_2 + R_4 + R_5)i_2 - R_5 i_3 \\
0 &= -R_5 i_2 + (R_3 + R_5)i_3
\end{aligned}
$$

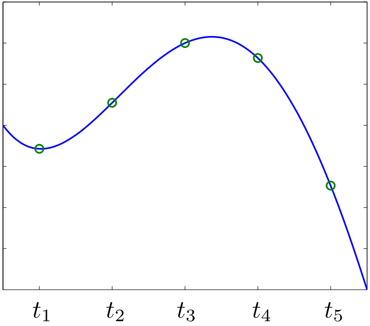
by KVL, we obtain a set of linear equations

Example: polynomial interpolation

fit a polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

through n points $(t_1, y_1), \ldots, (t_n, y_n)$



write out the conditions on x:

$$\begin{aligned}
p(t_1) &= x_1 + x_2 t_1 + x_3 t_1^2 + \cdots + x_n t_1^{n-1} \\
p(t_2) &= x_1 + x_2 t_2 + x_3 t_2^2 + \cdots + x_n t_2^{n-1} \\
&\vdots \\
p(t_n) &= x_1 + x_2 t_n + x_3 t_n^2 + \cdots + x_n t_n^{n-1}
\end{aligned}$$

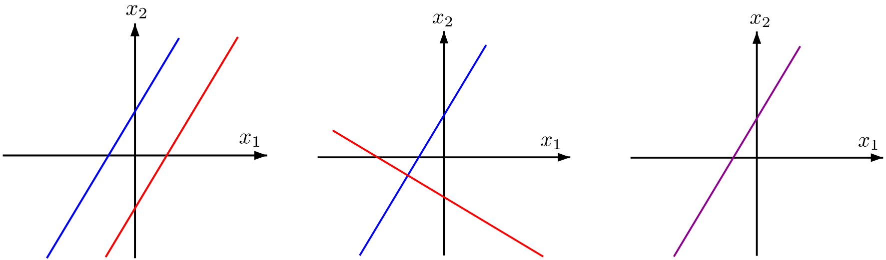
problem data (parameters): $(t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)$

problem variables: find $x_1, \ldots, x_n$ such that $p(t_i) = y_i$ for all i

Special case: two variables

Examples:

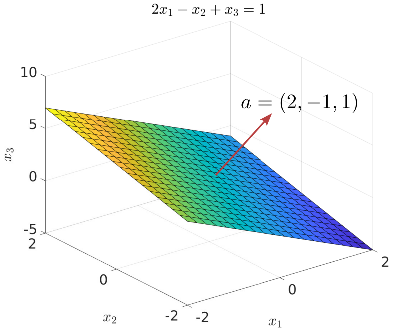
$$\begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= 2 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ x_1 + x_2 &= -1 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= -2 \end{aligned}$$



(a) no solution (b) one solution (c) many solutions

- no solution if two lines are parallel but different interceptions on x_2-axis
- many solutions if the two lines are identical

Geometrical interpretation



the set of solutions to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

can be interpreted as a hyperplane on $\mathbf{R}^n$

a solution to m linear equations is an **intersection** of m hyperplanes

Three types of linear equations

- **square** if $m = n$ (A is square)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **underdetermined** if $m < n$ (A is fat)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **overdetermined** if $m > n$ (A is skinny)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Existence and uniqueness of solutions

given a system of linear equations **existence:**

- no solution (the linear system is **inconsistent**)
- a solution exists (the linear system is **consistent**)

  uniqueness:

  - the solution is unique
  - there are infinitely many solutions

every system of linear equations has zero, one, or infinitely many solutions

there are no other possibilities

no solution

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= 2 \end{aligned}$$

unique solution

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 0 \end{aligned} \Rightarrow x = (1/3, 2/3) \qquad \begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= -2 \end{aligned} \Rightarrow x = (-1, 1)$$

infinitely many solutions

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + x_3 &= -1 \\ 3x_1 - 2x_2 + 3x_3 &= 3 \end{aligned}$$

$$x = (1 - t, t), \qquad x = (1 - t, 3t, t), \qquad t \in \mathbf{R}$$

Elementary row operations

define the **augmented matrix** of the linear equations on page 5 as

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

the following operations on the row of the augmented matrix:

1. multiply a row through by a nonzero constant
2. interchange two rows
3. add a constant times one row to another

do not alter the solution set and yield a simpler system

these are called **elementary row operations** on a matrix

Example

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \overset{\text{augmented matrix}}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add the first row to the second ($R_1 + R_2 \to R_2$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add -2 times the first row to the third ($-2R_1 + R_3 \to R_3$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ -7x_2 - 6x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

multiply the second row by $1/4$ ($R_2/4 \to R_2$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ -7x_2 - 6x_3 & = & -1 \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

add 7 times the second row to the third ($7R_2 + R_3 \to R_3$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ -\frac{3}{4}x_3 & = & \frac{3}{4} \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & -3/4 & 3/4 \end{bmatrix}$$

multiply the third row by $-4/3$ ($-4R_3/3 \to R_3$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ x_3 & = & -1 \end{array} \implies \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add $-3/4$ times the third row to the second $(R_2 - (3/4)R_3 \rightarrow R_2)$

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add -3 times the second row to the first $(R_1 - 3R_2 \rightarrow R_1)$

$$\begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add -2 times the third row to the first $(R_1 - 2R_2 \rightarrow R_1)$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Gaussian elimination

- a systematic procedure for solving systems of linear equations
- based on performing row operations of the augmented matrix
- simplifies the system of equations into an easy form where a solution can be obtained by inspection

Row echelon form

definition: a matrix is in **row echelon form** if

1. a row does not consist entirely of zeros, then the first nonzero number in the row is a 1 (called a **leading 1**)
2. all nonzero rows are above any rows of all zeros
3. in any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row

examples:

$$\begin{bmatrix} 1 & 4 & -3 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Reduced row echelon form

definition: a matrix is in **reduced row echelon form** if

- it is in a row echelon form and
- every leading 1 is the only nonzero entry in its column

examples:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Facts about echelon forms

1. every matrix has a *unique* reduced row echelon form
2. row echelon forms are not unique

$$\text{example:} \quad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3. all row echelon forms of a matrix have the same number of zero rows
4. the leading 1's always occur in the same positions in the row echelon forms of a matrix A
5. the columns that contain the leading 1's are called **pivot columns** of A
6. **rank** of A is defined as

$$\text{the number of nonzero rows of (reduced) echelon form of } A$$

Inspecting a solution

- simplify the augmented matrix to the *reduced echelon form*
- read the solution from the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies 0 \cdot x_3 = 1 \quad \text{(no solution)}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \implies x_1 = -2, \;\; x_2 = -1, \;\; x_3 = 5 \quad \text{(unique solution)}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies x_1 = 2, \; x_2 = 1 \quad \text{(unique solution)}$$

Leading and free variables

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{aligned} x_1 + 3x_2 &= -2 \\ x_2 - x_3 &= 1 \end{aligned}$$

definition:

- the corresponding variables to the leading 1's are called **leading variables**
- the remaining variables are called **free variables**

here x_1, x_2 are leading variables and x_3 is a free variable

let $x_3 = t$ and we obtain

$$x_1 = -3t - 2, \quad x_2 = t + 1, \quad x_3 = t$$

(many solutions)

General solution

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies x_1 - 5x_2 + x_3 = 4$$

x_1 is the leading variable, x_2 and x_3 are free variables

let $x_2 = s$ and $x_3 = t$ we obtain

$$\begin{aligned} x_1 &= 5s - t + 4 \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

(many solutions)

by assigning values to s and t, a set of parametric equations:

$$\begin{aligned} x_1 &= 5s - t + 4 \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

is called a **general solution** of the system

Solution to a linear system

solving $b = Ax$ with $A \in \mathbf{R}^{m \times n}$ has only three possibilities

1. **no solution**: if $\mathbf{rank}([A|b]) \neq \mathbf{rank}(A)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right]$$

2. **unique solution**: if $\mathbf{rank}([A|b]) = \mathbf{rank}(A) = n$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 2 & 3 \end{array}\right]$$

3. **infinitely many solution**: if $\mathbf{rank}([A|b]) = \mathbf{rank}(A) < n$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

Gaussian-Jordan elimination

- simplify an augmented matrix to the reduced row echelon form
- inspect the solution from the reduced row echelon form
- the algorithm consists of two parts:
  - **forward phase:** zeros are introduced below the leading 1's
  - **backward phase:** zeros are introduced above the leading 1's

Example

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 & = & 8 \\ -x_1 - 2x_2 + 3x_3 & = & 1 \\ 3x_1 - 7x_2 + 4x_3 & = & 10 \end{array} \implies \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

use row operations

$$\begin{array}{ccc} R_1 + R_2 \to R_2 & -3R_1 + R_3 \to R_3 & (-1) \cdot R_2 \to R_2 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} 10R_2 + R_3 \to R_3 & R_3/(-52) \to R_3 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

(a row echelon form)

we have added zero below the leading 1's (forward phase) continue performing row operations

$$\begin{array}{ccc} 5R_3 + R_2 \rightarrow R_2 & -R_2 + R_1 \rightarrow R_1 & -2R_3 + R_1 \rightarrow R_1 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

(reduced echelon form)

we have added zero above the leading 1's (backward phase)

from the reduced echelon form, $\mathbf{rank}([A|b]) = \mathbf{rank}(A) = n$

the system has a unique solution

$$x_1 = 3, \quad x_2 = 1, \quad x_3 = 2$$

Homogeneous linear systems

definition:

a system of linear equations is said to be **homogeneous** if b_j's are all zero

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \quad &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}$$

- $x_1 = x_2 = \cdots = x_n = 0$ is the **trivial** solution to $Ax = 0$
- if $(x_1, x_2, \ldots, x_n)$ is a solution, so is $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ for any $\alpha \in \mathbf{R}$
- hence, if a solution exists, then the system has infinitely many solutions (by choosing α arbitrarily)
- if z and w are solutions to $Ax = 0$, so is $z + \alpha w$ for any $\alpha \in \mathbf{R}$

example

$$\begin{array}{rcl} x_1 - x_2 + 2x_3 - x_4 & = & 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 & = & 0 \\ -x_1 + 2x_2 - 4x_3 + x_4 & = & 0 \\ 3x_1 - 3x_4 & = & 0 \end{array} \implies \begin{bmatrix} 1 & -1 & 2 & -1 & 0 \\ 2 & 1 & -2 & -2 & 0 \\ -1 & 2 & -4 & 1 & 0 \\ 3 & 0 & 0 & -3 & 0 \end{bmatrix}$$

the reduced echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{rcl} x_1 - x_4 & = & 0 \\ x_2 - 2x_3 & = & 0 \end{array}$$

define $x_3 = s, x_4 = t$, the parametric equation is

$$x_1 = t, \quad x_2 = 2s, \quad x_3 = s, \quad x_4 = t$$

there are two nonzero rows, so we have two $(n - 2 = 2)$ free variables

Properties of homogeneous linear system

more properties:

- the last column of the augmented matrix is entirely zero (and hence, can be neglected in the augmented matrix)
- if the reduced row echelon form has r *nonzero* rows, then the system has $n - r$ free variables
- a homogeneous linear system with more unknowns than equations has infinitely many solutions

Range space of A

range space of $A \in \mathbf{R}^{m \times n}$ is

$$\begin{aligned} \mathcal{R}(A) &= \{\, y \in \mathbf{R}^m \mid y = Ax, \ \text{ for } x \in \mathbf{R}^n \,\} \\ \mathbf{rank}(A) &\triangleq \text{number of leading 1's in row echelon form of } A \end{aligned}$$

- $y \in \mathcal{R}(A)$ if and only if y is a linear combination of columns in A:

$$y = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

- a linear system $y = Ax$ has a solution if and only if $y \in \mathcal{R}(A)$ (existence)
- equivalently, $y = Ax$ has a solution if and only if $\mathbf{rank}(A) = \mathbf{rank}([A \mid y])$

Nullspace of A

nullspace of A is

$$\mathcal{N}(A) = \{ \, x \in \mathbf{R}^n \mid Ax = 0 \, \}$$

example:

$$A = \begin{bmatrix} 2 & -5 & 3 & 0 \\ -2 & -1 & 3 & -1 \\ 5 & -1 & -3 & 2 \end{bmatrix}, \quad \Longrightarrow R = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1 & 1/12 \end{bmatrix}, \quad x = x_4 \begin{bmatrix} -1/2 \\ -1/4 \\ -1/12 \\ 1 \end{bmatrix}, x_4 \in \mathbf{R}$$

uniqueness of solution:

- if the linear system has a solution, the solution is unique if and only if $\mathcal{N}(A) = \{0\}$
- if x_p is a solution to $Ax = b$, and $\mathcal{N}(A) \neq \{0\}$ then a general solution to $Ax = b$ can be expressed as $x = x_p + z$ where $z \in \mathcal{N}(A)$ (infinitely many solutions)

Summary of solving linear systems

for $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^{m \times n}$, the linear system $Ax = b$ has a solution if and only if

$$b \in \mathcal{R}(A) \quad \Longleftrightarrow \quad \mathbf{rank}([A|b]) = \mathbf{rank}(A)$$

if $Ax = b$ has a solution, the uniqueness of the solution in three cases:

- **square A:** the solution is unique $\Leftrightarrow \mathcal{N}(A) \neq \{0\} \Leftrightarrow$ no zero rows in reduced echelon form of A
- **tall A:** the solution is unique $\Leftrightarrow \mathcal{N}(A) \neq \{0\}$
- **fat A:** since $\mathcal{N}(A) \neq \{0\}$ (always), the solutions are never unique

References

1. W.K. Nicholson, *Linear Algebra with Applications*, McGraw-Hill, 2006
2. H.Anton and C. Rorres, *Elementary Linear Algebra*, John Wiley, 2011
3. S. Boyd and L. Vandenberghe, *Introduction to Applied Linear Algebra: Vectors, Matrices, and Least squares*, Cambridge, 2018