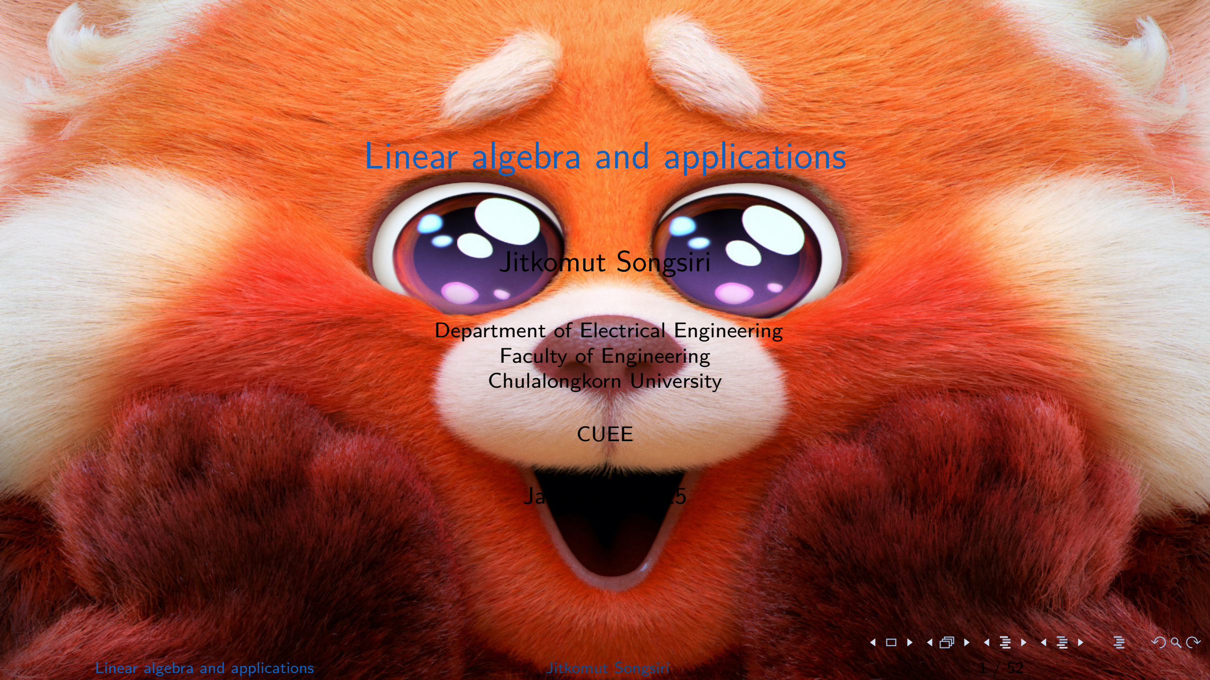

Linear algebra and applications

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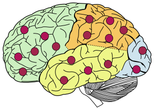
Outline

1. Eigenvalues and eigenvectors

How to read this handout

1. the note is used with lecture in EE205 (you cannot master this topic just by reading this note) – class activities include
   - graphical concepts, math derivation of details/steps in between
   - computer codes to illustrate examples
2. always read 'textbooks' after lecture
3. pay attention to the symbol ✎; you should be able to prove such ✎ result
4. each chapter has a list of references; find more formal details/proofs from in-text citations
5. almost all results in this note can be Googled; readers are encouraged to 'stimulate neurons' in your brain by proving results without seeking help from the Internet first
6. typos and mistakes can be reported to `jitkomut@gmail.com`



Eigenvalues and eigenvectors

Linear Independence

Definition: a set of vectors $\{v_1, v_2, \dots, v_n\}$ is **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \Longrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

equivalent conditions:

- coefficients of $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ are uniquely determined, i.e.,

  $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$$

  implies $\alpha_k = \beta_k$ for $k = 1, 2, \dots, n$
- no vector v_i can be expressed as a linear combination of the other vectors

Examples

- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}$ are independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$ are independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}4\\2\\0\end{bmatrix}$ are not independent
- $\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ are not independent

Linear span

Definition: the linear span of a set of vectors

$$\{v_1, v_2, \ldots, v_n\}$$

is the set of all linear combinations of $v_1, \ldots, v_n$

$$\mathrm{span}\{v_1, v_2, \ldots, v_n\} = \{a_1 v_1 + a_2 v_2 + \cdots + a_n v_n \mid a_1, \ldots, a_n \in \mathbf{R}\}$$

example:

$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is the hyperplane on $x_1 x_2$ plane

Eigenvalues

$\lambda \in \mathbf{C}$ is called an **eigenvalue** of $A \in \mathbf{C}^{n\times n}$ if

$$\det(\lambda I - A) = 0$$

equivalent to:

- there exists nonzero $x \in \mathbf{C}^n$ s.t. $(\lambda I - A)x = 0$, *i.e.*,

  $$Ax = \lambda x$$

  any such x is called an **eigenvector** of A (associated with eigenvalue λ)
- there exists nonzero $w \in \mathbf{C}^n$ such that

  $$w^T A = \lambda w^T$$

  any such w is called a **left eigenvector** of A

Computing eigenvalues

- $\mathcal{X}(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of A
- $\mathcal{X}(\lambda) = 0$ is called the **characteristic equation** of A
- eigenvalues of A are the root of characteristic polynomial

Computing eigenvalues

- $\mathcal{X}(\lambda) = \det(\lambda I - A)$ is called the **characteristic polynomial** of A
- $\mathcal{X}(\lambda) = 0$ is called the **characteristic equation** of A

the characteristic equation provides a way to compute the eigenvalues of A

$$A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$$

$$\mathcal{X}(\lambda) = \begin{vmatrix} \lambda - 5 & -3 \\ 6 & \lambda + 4 \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$$

solving the characteristic equation gives

$$\lambda = 2, -1$$

Computing eigenvectors

for each eigenvalue of A, we can find an associated eigenvector from

$$(\lambda I - A)x = 0$$

where x is a **nonzero** vector

for A in page 10, let's find an eigenvector corresponding to $\lambda = 2$

$$(\lambda I - A)x = \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad x_1 + x_2 = 0$$

the equation has many solutions, so we can form the set of solutions by

$$\left\{ x \in \mathbf{R}^2 \;\middle|\; x = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

this set is called the **eigenspace** of A corresponding to $\lambda = 2$

Eigenspace

eigenspace of A corresponding to λ is defined as the nullspace of $\lambda I - A$

$$\mathcal{N}(\lambda I - A)$$

equivalent definition: solution space of the homogeneous system

$$(\lambda I - A)x = 0$$

- an eigenspace is a vector space (by definition)
- 0 is in every eigenspace but it is not an eigenvector
- the *nonzero* vectors in an eigenspace are the eigenvectors of A

from page 11, any nonzero vector lies in the eigenspace is an eigenvector of A, *e.g.*, $x = \begin{bmatrix} -1 & 1 \end{bmatrix}^T$

same way to find an eigenvector associated with $\lambda = -1$

$$(\lambda I - A)x = \begin{bmatrix} -6 & -3 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad 2x_1 + x_2 = 0$$

so the eigenspace corresponding to $\lambda = -1$ is

$$\left\{ x \;\middle|\; x = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} \right\} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

and $x = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$ is an eigenvector of A associated with $\lambda = -1$

Properties

- if A is $n \times n$ then $\mathcal{X}(\lambda)$ is a polynomial of order n
- if A is $n \times n$ then there are n eigenvalues of A
- even when A is real, eigenvalues and eigenvectors can be complex, *e.g.*,

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$$

- if A and λ are real, we can choose the associated eigenvector to be real
- if A is real then eigenvalues must occur in complex conjugate pairs
- if x is an eigenvector of A, so is αx for any $\alpha \in \mathbf{C}$, $\alpha \neq 0$
- an eigenvector of A associated with λ lies in $\mathcal{N}(\lambda I - A)$

Important facts

denote $\lambda(A)$ an eigenvalue of A

- $\lambda(\alpha A) = \alpha\lambda(A)$ for any $\alpha \in \mathbf{C}$
- $\mathbf{tr}(A)$ is the sum of eigenvalues of A
- $\det(A)$ is the product of eigenvalues of A
- A and A^T share the same eigenvalues ✎
- $\lambda(\overline{A^T}) = \overline{\lambda(A)}$ ✎
- $\lambda(A^m) = (\lambda(A))^m$ for any integer m
- A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A ✎

Matrix powers

the mth power of a matrix A for a nonnegative integer m is defined as

$$A^m = \prod_{k=1}^{m} A$$

(the multiplication of m copies of A)

and A^0 is defined as the identity matrix, *i.e.*, $A^0 = I$

✌ **Facts:** if λ is an eigenvalue of A with an eigenvector v then

- λ^m is an eigenvalue of A^m
- v is an eigenvector of A^m associated with λ^m

Invertibility and eigenvalues

A is not invertible if and only if there exists a nonzero x such that

$$Ax = 0, \quad \text{or} \quad Ax = 0 \cdot x$$

which implies 0 is an eigenvalue of A

another way to see this is that

$$A \text{ is not invertible} \iff \det(A) = 0 \iff \det(0 \cdot I - A) = 0$$

which means 0 is a root of the characteristic equation of A

conclusion ✎ the following statements are equivalent

- A is invertible
- $\mathcal{N}(A) = \{0\}$
- $\lambda = 0$ is not an eigenvalue of A

Eigenvalues of special matrices

diagonal matrix:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

eigenvalues of D are the diagonal elements, *i.e.*, $\lambda = d_1, d_2, \ldots, d_n$

triangular matrix:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \quad L = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

eigenvalues of L and U are the diagonal elements, *i.e.*, $\lambda = a_{11}, \ldots, a_{nn}$

Similarity transform

two $n \times n$ matrices A and B are said to be **similar** if

$$B = T^{-1}AT$$

for some invertible matrix T

T is called a **similarity transform**

✌ **invariant** properties under similarity transform:

- $\det(B) = \det(A)$
- $\mathbf{tr}(B) = \mathbf{tr}(A)$
- A and B have the same eigenvalues

$$\det(\lambda I - B) = \det(\lambda T^{-1}T - T^{-1}AT) = \det(\lambda I - A)$$

Diagonalization

an $n \times n$ matrix A is **diagonalizable** if there exists T such that

$$T^{-1}AT = D$$

is *diagonal*

- similarity transform by T *diagonalizes* A
- A and D are similar, so the entries of D must be the eigenvalues of A

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

- computing A^k is simple because $A^k = (TDT^{-1})^k = TD^kT^{-1}$

Eigenvalue decomposition

if A is diagonalizable then A admits the decomposition

$$A = TDT^{-1}$$

- D is diagonal containing the eigenvalues of A
- columns of T are the corresponding eigenvectors of A
- note that such decomposition is not unique (up to scaling in T)

Theorem: $A \in \mathbf{R}^{n \times n}$ is diagonalizable if and only if all n eigenvectors of A are independent

- a diagonalizable matrix is called a **simple** matrix
- if A is not diagonalizable, sometimes it is called *defective*

Proof (necessity)

suppose $\{v_1, \dots, v_n\}$ is a *linearly independent* set of eigenvectors of A

$$Av_i = \lambda_i v_i \quad i = 1, \dots, n$$

we can express this equation in the matrix form as

$$A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

define $T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ and $D = \mathbf{diag}(\lambda_1, \dots, \lambda_n)$, so

$$AT = TD$$

since T is invertible ($v_1, \dots, v_n$ are independent), finally we have

$$T^{-1}AT = D$$

Proof (sufficiency)

conversely, if there exists $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$ that diagonalizes A

$$T^{-1}AT = D = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

then $AT = TD$, or

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

so $\{v_1, \ldots, v_n\}$ is a linearly independent set of eigenvectors of A

Example

find T that diagonalizes

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

the characteristic equation is

$$\det(\lambda I - A) = \lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

the eigenvalues of A are $\lambda = 5, 3, 3$
an eigenvector associated with $\lambda_1 = 5$ can be found by

$$(5 \cdot I - A)x = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \\ x_3 \text{ is a free variable} \end{array}$$

an eigenvector is $v_1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$

next, find an eigenvector associated with $\lambda_2 = 3$

$$(3 \cdot I - A)x = \begin{bmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_1 + x_3 = 0 \\ x_2, x_3 \text{ are free variables} \end{array}$$

the eigenspace can be written by

$$\left\{ x \;\middle|\; x = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

hence we can find two *independent* eigenvectors

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

corresponding to the repeated eigenvalue $\lambda_2 = 3$

easy to show that v_1, v_2, v_3 are linearly independent

we form a matrix T whose columns are v_1, v_2, v_n

$$T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

then v_1, v_2, v_3 are linearly independent if and only if T is invertible

by a simple calculation, $\det(T) = 2 \neq 0$, so T is invertible

hence, we can use this T to diagonalize A and it is easy to verify that

$$T^{-1}AT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Not all matrices are diagonalizable

example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

characteristic polynomial is $\det(\lambda I - A) = s^2$, so 0 is the only eigenvalue

eigenvector satisfies $Ax = 0 \cdot x$, *i.e.*,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad \Longrightarrow \quad \begin{array}{c} x_2 = 0 \\ x_1 \text{ is a free variable} \end{array}$$

so all eigenvectors has form $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$ where $x_1 \neq 0$

thus A cannot have *two independent* eigenvectors

Distinct eigenvalues

Theorem: if A has distinct eigenvalues, *i.e.*,

$$\lambda_i \neq \lambda_j, \quad i \neq j$$

then a set of corresponding eigenvectors are *linearly independent*

which further implies that A is diagonalizable

the converse is *false* – A can have repeated eigenvalues but still be diagonalizable

example: all eigenvalues of I are 1 (repeated eigenvalues) but I is diagonal

Proof by contradiction

assume the eigenvectors are dependent

(simple case) let $Ax_k = \lambda_k x_k$, $k = 1, 2$

suppose there exists $\alpha_1, \alpha_2 \neq 0$

$$\alpha_1 x_1 + \alpha_2 x_2 = 0 \qquad (1)$$

multiplying (1) by A: $\quad \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 = 0$

multiplying (1) by λ_1: $\quad \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_1 x_2 = 0$

subtracting the above from the previous equation

$$\alpha_2 (\lambda_2 - \lambda_1) x_2 = 0$$

since $\lambda_1 \neq \lambda_2$, we must have $\alpha_2 = 0$ and consequently $\alpha_1 = 0$

the proof for a general case is left as an exercise

Algebraic and Geometric multiplicities

algebraic multiplicity of an eigenvalue λ_k is defined as the multiplicity of the root λ_k of the characteristic polynomial

example: the characteristic polynomial of A is

$$\mathcal{X}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)^2(\lambda - \lambda_3)^5$$

the multiplicity of λ_1, λ_2 and λ_3 are $1, 2$ and 5 respectively

geometric multiplicity of an eigenvalue λ_k is defined as

$$\dim \mathcal{N}(\lambda_k I - A)$$

(the dimension of the corresponding eigenspace)

example: $A = I_n$; the geometric multiplicity of 1 is n

let λ be an eigenvalue of a matrix A $(n \times n)$

Theorem ✌

- the geometric multiplicity of λ is the number of linearly independent eigenvectors associated with λ
- algebraic and geometric multiplicities need not be equal
- let r be the algebraic multiplicity of λ

$$\dim \mathcal{N}(\lambda I - A) \leq r$$

  (the geometric multiplicity is less than or equal to the algebraic multiplicity)
- A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity

Matrix Power

the mth power of a matrix A for a *nonnegative* m is defined as

$$A^m = \prod_{k=1}^{m} A$$

and define $A^0 = I$

property: $A^r A^s = A^s A^r = A^{r+s}$

a *negative* power of A is defined as

$$A^{-n} = (A^{-1})^n$$

n is a nonnegative integer and A is invertible

Matrix polynomial

a **matrix polynomial** is a polynomial with matrices as variables

$$p(A) = a_0 I + a_1 A + \cdots + a_n A^n$$

for example $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$

$$\begin{aligned} p(A) = 2I - 6A + 3A^2 &= 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 6\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 2 & -3 \\ 0 & 11 \end{bmatrix} \end{aligned}$$

Fact ✎ any two polynomials of A commute, *i.e.*, $p(A)q(A) = q(A)p(A)$

Matrix exponential via diagonalization

suppose A is diagonalizable, *i.e.*, $\Lambda = T^{-1}AT \quad \Longleftrightarrow \quad A = T\Lambda T^{-1}$

where $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$, *i.e.*, the columns of T are eigenvectors of A

then we have $A^k = T\Lambda^k T^{-1}$

thus diagonalization simplifies the expression of a matrix polynomial

$$\begin{aligned} p(A) &= a_0 I + a_1 A + \cdots + a_n A^n \\ &= a_0 T T^{-1} + a_1 T\Lambda T^{-1} + \cdots + a_n T\Lambda^n T^{-1} \\ &= T p(\Lambda) T^{-1} \end{aligned}$$

where

$$p(\Lambda) = \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix}$$

Eigenvectors of matrix polynomial

if λ and v be an eigenvalue and corresponding eigenvector of A then

- $p(\lambda)$ is an eigenvalue of $p(A)$
- v is a corresponding eigenvector of $p(A)$

$$Av = \lambda v \quad \Longrightarrow \quad A^2 v = \lambda A v = \lambda^2 v \quad \cdots \quad \Longrightarrow \quad A^k v = \lambda^k v$$

thus

$$(a_0 I + a_1 A + \cdots + a_n A^n) v = (a_0 v + a_1 \lambda + \cdots + a_n \lambda^n) v$$

which shows that

$$p(A)v = p(\lambda)v$$

Matrix exponential

generalize the exponential function of a scalar

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

to an exponential function of a matrix

define **matrix exponential** as

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \quad = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

for a square matrix A

the infinite series converges for all A

Example

example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

find all powers of A

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ldots, \quad A^k = A \quad \text{for } k = 2, 3, \ldots$$

so by definition,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + \sum_{k=1}^{\infty} \frac{A^k}{k!} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

never compute e^A by element-wise operation !

$$e^A \neq \begin{bmatrix} e^1 & e^1 \\ e^0 & e^0 \end{bmatrix}$$

Eigenvalues of matrix exponential

✌ if λ and v be an eigenvalue and corresponding eigenvector of A then

- e^{λ} is an eigenvalue of e^{A}
- v is a corresponding eigenvector of e^{A}

since e^{A} can be expressed as power series of A:

$$e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

multiplying v on both sides and using $A^k v = \lambda^k v$ give

$$\begin{aligned} e^{A}v &= v + Av + \tfrac{A^2 v}{2!} + \tfrac{A^3 v}{3!} + \cdots \\ &= \left(1 + \lambda + \tfrac{\lambda^2}{2!} + \tfrac{\lambda^3}{3!} + \cdots\right) v \\ &= e^{\lambda} v \end{aligned}$$

Properties of matrix exponential

- $e^0 = I$
- $e^{A+B} \neq e^A \cdot e^B$
- if $AB = BA$, *i.e.*, A and B commute, then $e^{A+B} = e^A \cdot e^B$
- $(e^A)^{-1} = e^{-A}$

✌ these properties can be proved by the definition of e^A

Computing e^A via diagonalization

if A is diagonalizable, *i.e.*,

$$T^{-1}AT = \Lambda = \mathbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

where λ_k's are eigenvalues of A then e^A has the form

$$e^A = Te^{\Lambda}T^{-1}$$

- computing e^{Λ} is simple since Λ is diagonal
- one needs to find eigenvectors of A to form the matrix T
- the expression of e^A follows from

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{(T\Lambda T^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{T\Lambda^k T^{-1}}{k!} = Te^{\Lambda}T^{-1}$$

- if A is diagonalizable, so is e^A

Example

example: compute $f(A) = e^A$ given $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

the eigenvalues and eigenvectors of A are

$$\lambda_1 = 1,\ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2,\ v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_3 = 0,\ v_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

form $T = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$ and compute $e^A = Te^{\Lambda}T^{-1}$

$$e^A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} e & e^2 - e & (e^2 - 2e + 1)/2 \\ 0 & e^2 & (e^2 - 1)/2 \\ 0 & 0 & 1 \end{bmatrix}$$

Applications to ordinary differential equations

we solve the following first-order ODEs for $t \geq 0$ where $x(0)$ is given

scalar: $x(t) \in \mathbf{R}$ and $a \in \mathbf{R}$ is given

$$\dot{x}(t) = ax(t)$$

solution: $x(t) = e^{at}x(0)$, for $t \geq 0$

vector: $x(t) \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$ is given

$$\dot{x}(t) = Ax(t)$$

solution: $x(t) = e^{At}x(0)$, for $t \geq 0$ (use $\dfrac{de^{At}}{dt} = Ae^{At} = e^{At}A$)

Applications to difference equations

we solve the difference equations for $t = 0, 1, \ldots$ where $x(0)$ is given

scalar: $x(t) \in \mathbf{R}$ and $a \in \mathbf{R}$ is given

$$x(t+1) = ax(t)$$

solution: $x(t) = a^t x(0)$, for $t = 0, 1, 2, \ldots$

vector: $x(t) \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$ is given

$$x(t+1) = Ax(t)$$

solution: $x(t) = A^t x(0)$, for $t = 0, 1, 2, \ldots$

Example 1

solve the ODE

$$\ddot{y}(t) - \dot{y}(t) - 6y(t) = 0, \quad y(0) = 1, \dot{y}(0) = 0$$

solution: define

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}$$

write the equation into the vector form $\dot{x}(t) = Ax(t)$

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \begin{bmatrix} \dot{y}(t) \\ \dot{y}(t) + 6y(t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ 6x_1(t) + x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x(t) \end{aligned}$$

the initial condition is

$$x(0) = \begin{bmatrix} y(0) \\ \dot{y}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example 1

thus it is left to compute e^{At}

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

the eigenvalues and eigenvectors of A are

$$\lambda_1 = -2,\ v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_2 = 3,\ v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

all eigenvalues are distinct, so A is diagonalizable and

$$e^{At} = Te^{\Lambda t}T^{-1}, \quad T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

Example 1

the closed-form expression of e^{At} is

$$e^{At} = \frac{1}{5}\begin{bmatrix} 3e^{-2t} + 2e^{3t} & -e^{-2t} + e^{3t} \\ -6e^{-2t} + 6e^{3t} & 2e^{-2t} + 3e^{3t} \end{bmatrix}$$

the solution to the vector equation is

$$\begin{aligned} x(t) = e^{At}x(0) &= \frac{1}{5}\begin{bmatrix} 3e^{-2t} + 2e^{3t} & -e^{-2t} + e^{3t} \\ -6e^{-2t} + 6e^{3t} & 2e^{-2t} + 3e^{3t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3e^{-2t} + 2e^{3t} \\ -6e^{-2t} + 6e^{3t} \end{bmatrix} \end{aligned}$$

hence the solution $y(t)$ can be obtained by

$$y(t) = x_1(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) = \frac{1}{5}\left(3e^{-2t} + 2e^{3t}\right), \quad t \geq 0$$

Example 2

solve the difference equation

$$y(t+2) - y(t+1) - 6y(t) = 0, \quad y(0) = 1, y(1) = 0$$

solution: define

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} y(t) \\ y(t+1) \end{bmatrix}$$

write the equation into the vector form $x(t+1) = Ax(t)$

$$\begin{aligned} x(t+1) &= \begin{bmatrix} y(t+1) \\ y(t+2) \end{bmatrix} = \begin{bmatrix} y(t+1) \\ y(t+1) + 6y(t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ 6x_1(t) + x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x(t) \end{aligned}$$

the initial condition is

$$x(0) = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Example 2

thus it is left to compute A^t

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

the eigenvalues and eigenvectors of A are

$$\lambda_1 = -2, \; v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \lambda_2 = 3, \; v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

all eigenvalues are distinct, so A is diagonalizable and

$$A^t = T\Lambda^t T^{-1}, \quad T = \begin{bmatrix} v_1 & v_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} (-2)^t & 0 \\ 0 & 3^t \end{bmatrix} \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$$

Example 2

the closed-form expression of A^t is

$$A^t = \frac{1}{5}\begin{bmatrix} 2(3^t) + 3(-2)^t & 3^t - (-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} & 3^{t+1} - (-2)^{t+1} \end{bmatrix}$$

for $t = 0, 1, 2, \ldots$

the solution to the vector equation is

$$\begin{aligned} x(t) = A^t x(0) &= \frac{1}{5}\begin{bmatrix} 2(3^t) + 3(-2)^t & 3^t - (-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} & 3^{t+1} - (-2)^{t+1} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 2(3^t) + 3(-2)^t \\ 2(3^{t+1}) + 3(-2)^{t+1} \end{bmatrix} \end{aligned}$$

hence the solution $y(t)$ can be obtained by

$$y(t) = x_1(t) = \frac{1}{5}\left(2(3^t) + 3(-2)^t\right), \quad t = 0, 1, 2, \ldots$$

Softwares (MATLAB)

1. `[V,D] = eig(A)` produces a diagonal matrix `D` of eigenvalues and a full matrix `V` whose columns are the corresponding eigenvectors
   - the eigenvectors are normalized to have a unit 2-norm
   - eigenvalues are not necessarily sorted by magnitude
2. eigs(A) returns the 6 largest magnitude eigenvalues
3. `expm(A)` computes the matrix exponential e^A
4. `exp(A)` computes the exponential of the entries in A

Softwares (Python)

1. `D,V = numpy.eig(A)` computes the eigenvalues and eigenvectors of A
2. `numpy.linalg.matrix_power(A, n)` computes the n power of A
3. `scipy.linalg.expm(A)` computes the matrix exponential of A
4. `numpy.exp(A)` computes the exponential of the entries of A

References

1. W.K. Nicholson, *Linear Algebra with Applications*, McGraw-Hill, 2006
2. H.Anton and C. Rorres, *Elementary Linear Algebra*, John Wiley, 2011