12. Series

- limit and convergence
- Taylor series
- Maclaurin series
- Laurent series

Convergence of sequences

an infinite **sequence**

$$z_1, z_2, \ldots, z_n, \ldots$$

of complex numbers has a **limit** z, denoted by

$$\lim_{n\to\infty} z_n = z$$

if for each $\epsilon > 0$, there exists a positive integer N such that

$$|z_n - z| < \epsilon \quad \text{whenver} \quad n > N$$

(z_n becomes arbitrarily close to z as n increases)

- if a limit exists it must be **unique**
- when the limit exists, the sequence is said to **converge to** z
- if the sequence has no limit, it **diverges**

Limit of complex-valued sequences

suppose that $z_n = x_n + jy_n$ and $z = x + jy$; then

$$\lim_{n\to\infty} z_n = z$$

if and only if

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y$$

example: $z_n = \frac{1}{n^3} + j$ for $n = 1, 2, \ldots$

$$\lim_{n\to} z_n = \lim_{n\to\infty} \frac{1}{n^3} + j \lim_{n\to\infty} 1 = 0 + j = j$$

moreover, we can see that for each $\epsilon > 0$

$$|z_n - j| = \frac{1}{n^3} \quad \text{whenver} \quad n > \frac{1}{\epsilon^{1/3}}$$

Convergence of series

an infinite **series**

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_k + \cdots$$

of complex numbers **converges** to the **sum** S if the sequence

$$S_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots + z_n \qquad (n = 1, 2, \ldots)$$

of **partial sums** converges to S; then we can write

$$\sum_{k=1}^{\infty} z_k = S \quad \text{if} \quad \lim_{n\to\infty} S_n = S$$

- a series can have **at most** one sum
- when a series does not converge, we say it **diverges**

Limit of complex-valued series

suppose that $z_n = x_n + jy_n$ and $S = X + jY$; then

$$\sum_{n=1}^{\infty} z_k = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \quad \text{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

Facts:

- if a series converges, the nth term converges to zero as $n \to \infty$
- the absolute convergence of a series implies the convergence of that series

$$\sum_{n=1}^{\infty} |z_n| \text{ converges} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} z_n \text{ converges}$$

example: the geometric series $\sum_{k=0}^{\infty} z^k$

the nth partial sum of the geometric series is given by

$$S_n = \sum_{k=0}^{n} z^k = 1 + z + z^2 + \cdots + z^{n-1} + z^n$$

multiply both sides by $1 - z$

$$(1-z)S_n = 1 - z + z - z^2 + \cdots + z^{n-1} - z^n + z^n - z^{n+1} = 1 - z^{n+1}$$

if $|z| < 1$ then $z^{n+1} \to 0$ and $S_n \to \dfrac{1}{1-z}$ as $n \to \infty$

the limit of the partial sum exists, and hence

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad |z| < 1$$

Taylor series

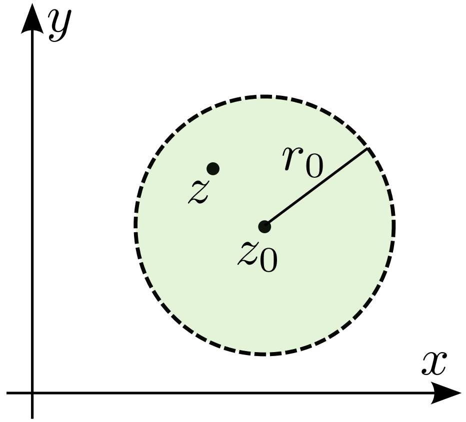
Taylor's theorem: suppose f is **analytic** throughout a disk $|z-z_0| < r_0$ then $f(z)$ has the *power series* representation

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)(z-z_0)^2}{2!} + \cdots + \frac{f^{(n)}(z_0)(z-z_0)^n}{n!} + \cdots$$

for each z inside the disk, $i.e.$, $|z-z_0| < r_0$

meaning: the power series converges to $f(z)$ when $|z-z_0| < r_0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$$



the expansion of $f(z)$ is called the **Taylor series** of f about the point z_0

Maclaurin series

when $z_0 = 0$, the Taylor series becomes

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!} \qquad (|z| < r_0)$$

and it is called a **Maclaurin series**

example: $f(z) = e^z$

since e^z is entire, it has a Maclaurin representation that is valid for all z

$$f^{(n)} = e^z, \ \ n = 0, 1, 2, \dots, \quad \Longrightarrow \quad f^{(n)}(0) = 1 \ \text{ for all } n$$

and it follows that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad (|z| < \infty)$$

example: Maclaurin representation of $f(z) = 1/(1-z)$

$f(z)$ is analytic throughout the open disk $|z| < 1$ and its derivatives are

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \quad \Longrightarrow \quad f^{(n)}(0) = n! \quad (n = 0, 1, 2, \ldots)$$

therefore, the Maclaurin series is given by

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

it is simply a **geometric series** where z is the common ratio of adjacent terms

agree with the result on page 12-6

example: Maclaurin representation of $f(z) = \sin z$

we write $\sin z = \dfrac{e^{jz} - e^{-jz}}{j2}$ and note that $\sin z$ is entire

then we can use the Maclaurin series of e^z for expanding $e^{\pm jz}$

$$\sin z = \frac{1}{j2}\left(\sum_{n=0}^{\infty}\frac{(jz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-jz)^n}{n!}\right) = \frac{1}{j2}\sum_{n=0}^{\infty}[1-(-1)^n]\frac{j^n z^n}{n!}$$

but $(1-(-1)^n) = 0$ when n is even and 2 otherwise, so we replace n by $2n+1$

$$\begin{aligned}\sin z &= \frac{1}{j2}\sum_{n=0}^{\infty}\frac{2j^{2n+1}z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \quad (|z| < \infty)\\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\end{aligned}$$

the series contains only **odd** powers of z

Maclaurin series expansion

for $|z| < \infty$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n+1)}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{(2n)}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

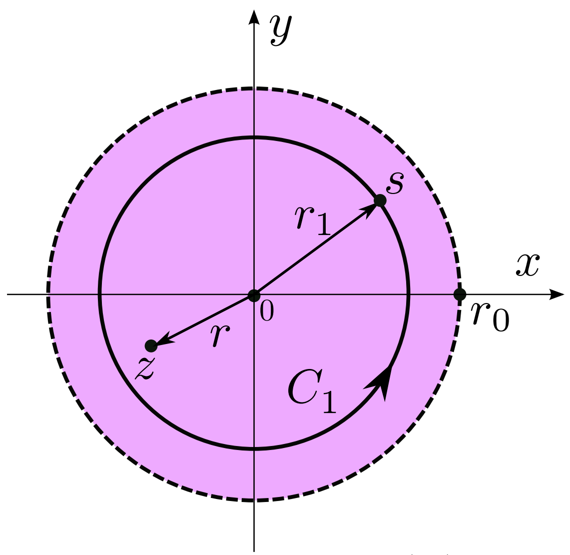
$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{(2n+1)}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{(2n)}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots$$

Proof of Taylor's theorem

assumption: f is analytic on $|z| < r_0$

proof for special case: $z_0 = 0$; $\quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!} \quad (|z| < r_0)$



- C_1 is a positively oriented circle $|z| = r_1$
- z is any point with $|z| = r$ and $r < r_1 < r_0$
- s is a point on contour C_1
- f is analytic *inside* and *on* the circle C_1

we will expand $f(z)$ from the Cauchy integral formula

$$f(z) = \frac{1}{j2\pi} \int_{C_1} \frac{f(s)}{s - z} ds$$

expand the integral term

- rewrite $1/(s-z)$ as $\dfrac{1}{s-z} = \dfrac{1}{s} \cdot \dfrac{1}{1-(z/s)}$
- for any $z \neq 1$,

$$\frac{1}{1-z} = \frac{z^N}{1-z} + \sum_{n=0}^{N-1} z^n \qquad \text{(from long division)}$$

- then we can write

$$\frac{1}{s-z} = \frac{z^N}{s^N(s-z)} + \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}}$$

- multiply by $f(s)$ and integrate with respect to s along C_1

$$\int_{C_1} \frac{f(s)}{s-z}\, ds = z^N \int_{C_1} \frac{f(s)}{(s-z)s^N}\, ds + \sum_{n=0}^{N-1} z^n \int_{C_1} \frac{f(s)}{s^{n+1}}\, ds$$

characterize the remainder term

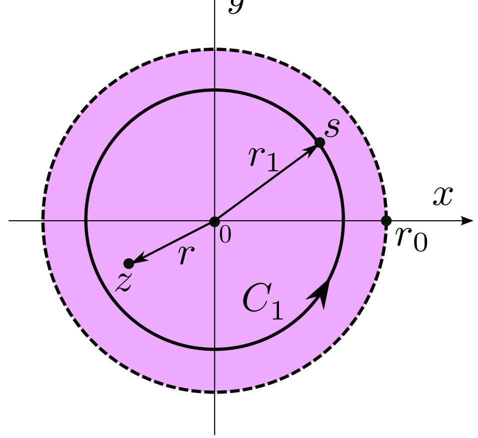
- the second term on RHS can be computed from the Cauchy integral formula

$$\int_{C_1} \frac{f(s)}{s^{n+1}} ds = j2\pi \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, 2, \ldots)$$

- from $f(z) = \frac{1}{j2\pi} \int_{C_1} \frac{f(s)}{s-z} ds$, we obtain

$$f(z) = \underbrace{\frac{z^N}{j2\pi} \int_{C_1} \frac{f(s)}{s^N (s-z)} \, ds}_{R_N(z)} + \sum_{n=0}^{N-1} \frac{f^{(n)}(0) z^n}{n!}$$

- we obtain Taylor's representation if we can show that $\lim_{N \to \infty} R_N(z) = 0$



the remainder term goes to zero as $n \to \infty$

- $|s - z| \geq ||s| - |z|| = r_1 - r$; hence $1/(s-z) \leq 1/(r_1 - r)$
- if $|f(s)| \leq M$ on C_1 then

$$|R_N(z)| = \left|\frac{z^N}{j2\pi}\right| \left|\int_{C_1} \frac{f(s)}{s^N(s-z)} ds\right| \leq \frac{r^N}{2\pi} \cdot \frac{M}{(r_1 - r)r_1^N} \cdot \underbrace{\text{length of } C_1}_{2\pi r_1}$$

$$= \left(\frac{r}{r_1}\right)^N \frac{M r_1}{(r_1 - r)} \to 0, \quad \text{as } N \to \infty \text{ because } r/r_1 < 1$$

- we finished the proof for the special case of Taylor's theorem; when $z_0 = 0$

generalize the result to $z_0 \neq 0$

assumption: f is analytic on $|z - z_0| < r_0$

- $f(z + z_0)$ must be analytic when $|(z + z_0) - z_0| < r_0$ (composite function)
- hence, $g(z) = f(z + z_0)$ is analytic on $|z| < r_0$, so its Maclaurin series is

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0) z^n}{n!} \qquad (|z| < r_0)$$

- this is equivalent to

$$f(z + z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0) z^n}{n!} \qquad (|z| < r_0)$$

- replace z by $z - z_0$, we obtain the Taylor's series

example: expand $f(z) = \dfrac{1+2z}{z^3+z^2}$ to a series involving powers of z

we cannot find a Maclaurin series for f since it is not analytic at $z=0$

however, for $|z| \neq 0$, we can write

$$\begin{aligned}
f(z) &= \frac{1}{z^2}\cdot\frac{1+2z}{1+z} = \frac{1}{z^2}\cdot\left(2-\frac{1}{1+z}\right) \\
&= \frac{1}{z^2}\cdot\big(2-(1-z+z^2-z^3+z^4-\cdots)\big) \qquad (|z|<1) \\
&= \frac{1}{z^2}(1+z-z^2+z^3-z^4+\cdots) \qquad (0<|z|<1) \\
&= \frac{1}{z^2}-\frac{1}{z}+z-z^2+\cdots
\end{aligned}$$

the expansion of f contains both *negative* and *positive* powers of z

remarks:

- if f fails to be analytic at a point z_0, we cannot apply Taylor's theorem there
- example in page 12-17 shows that however, it is possible to find a series for $f(z)$ involving both *positive* and *negative* powers of $(z - z_0)$

$$f(z) = \frac{1+2z}{z^3+z^2} = \frac{1}{z^2} - \frac{1}{z} + z - z^2 + \cdots$$

- such a representation is known as **Laurent series**, which includes the Taylor series as a special case
- with the Laurent series, we can expand f about a singular point

Laurent series

Theorem: if all of the following assumptions hold

1. D is an **annular** domain $r_1 < |z - z_0| < r_2$
2. C is any positively oriented simple closed contour around z_0 and lies inside D
3. f is analytic throughout D

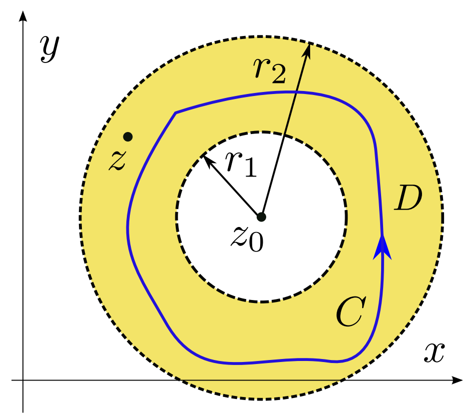
then f has the series representation; called the **Laurent series**

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad (n = 0, 1, \ldots)$$

$$b_n = \frac{1}{j2\pi} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz, \quad (n = 1, 2, \ldots)$$



remarks:

- we cannot apply the Cauchy integral formula to compute the coefficient a_n

$$a_n = \frac{1}{j2\pi}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

  because f is NOT analytic in C

- if the annular domain is specified, a Laurent series of $f(z)$ about z_0 is unique
- the annulus D is the region of convergence for the obtained Laurent series
- the coeff a_n and b_n given by the formula are generally difficult to compute
- so, we use another way such as computing a partial fraction of f and use

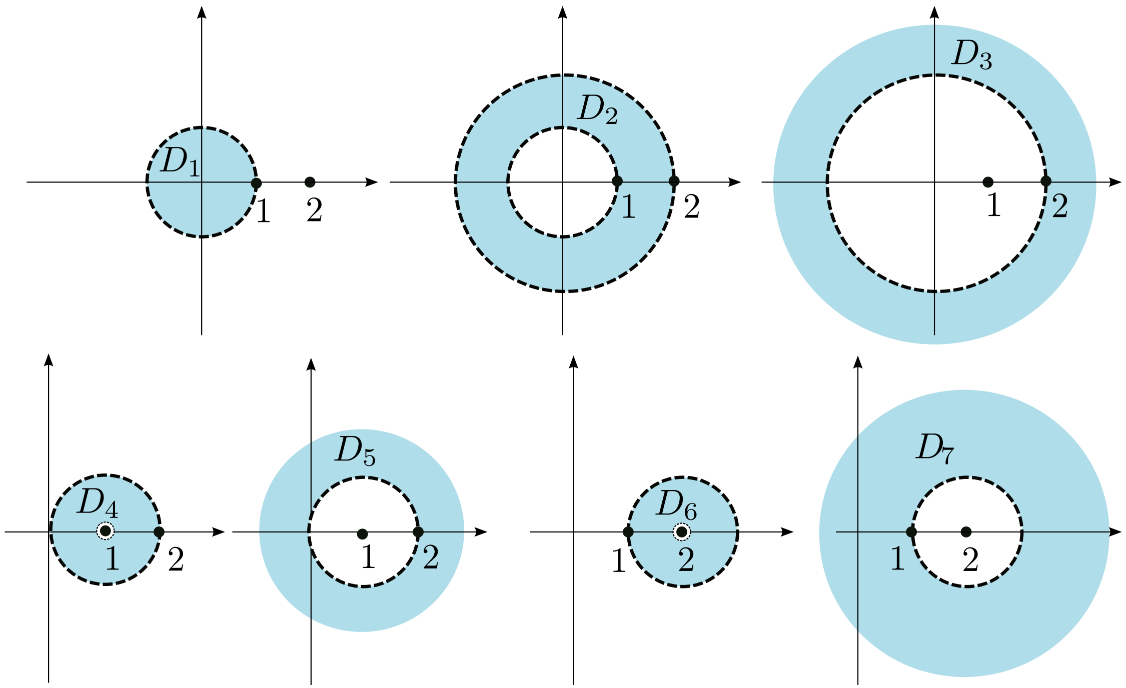
$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \qquad |z| < 1$$

  to expand the partial fraction as an infinite series

example: find power series representation of $f(z) = \dfrac{-1}{(z-1)(z-2)}$ in

$$D_1 : |z| < 1, \qquad D_2 : 1 < |z| < 2, \qquad D_3 : 2 < |z| < \infty$$

$$D_4 : 0 < |z-1| < 1, \quad D_5 : 1 < |z-1|, \quad D_6 : 0 < |z-2| < 1, \quad D_7 : 1 < |z-2|$$



f is not analytic at $z = 1$ and $z - 2$

- **domain** D_1: $|z| < 1$ $\qquad$ ($|z| < 1$ and $|z/2| < 1$ for all $z \in D_1$)

$$f(z) = f(z) = \frac{-1}{1-z} + \frac{1/2}{1-(z/2)}$$

$$= -\sum_{n=0}^{\infty} z^n + (1/2)\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}(2^{-n-1}-1)z^n, \qquad |z| < 1$$

the representation is a Maclaurin series

- **domain** D_2: $1 < |z| < 2$ $\qquad$ ($|1/z| < 1$ and $|z/2| < 1$ for all $z \in D_2$)

$$f(z) = \frac{1}{z}\cdot\frac{1}{1-(1/z)} + \frac{1}{2}\cdot\frac{1}{1-(z/2)}$$

$$= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \qquad (1 < |z| < 2)$$

this is *the* Laurent series for f in D_2 where $a_n = 1/2^{n+1}$ and $b_n = 1$

- **domain** D_3**:** $2 < |z| < \infty$ ($|2/z| < 1$ and so $|1/z| < 1$ for all $z \in D_3$)

$$f(z) = \frac{1}{z} \cdot \frac{1}{1-(1/z)} - \frac{1}{z} \cdot \frac{1}{1-(2/z)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=1}^{\infty} \frac{(1-2^{n-1})}{z^n}, \qquad (2 < |z| < \infty)$$

this is *the* Laurent series for f in D_3 where $a_n = 0$ and $b_n = 1 - 2^{n-1}$

- **domain** D_4**:** $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-1} + \frac{1}{1-(z-1)}$$

$$= \frac{1}{z-1} + \sum_{n=0}^{\infty} (z-1)^n \quad (0 < |z-1| < 1)$$

this is *the* Laurent series for f in D_4 where $b_1 = 1, b_k = 0, k \geq 2$ and $a_n = 1$

- **domain** D_5**:** $1 < |z-1|$ $\qquad$ ($1/|z-1| < 1$ for all $z \in D_5$)

$$f(z) = \frac{-1}{(z-1)(z-1-1)} = \frac{-1}{(z-1)^2} \cdot \frac{1}{1-1/(z-1)}$$
$$= -\sum_{n=0}^{\infty} \frac{1}{(z-1)^{n+2}}, \qquad (1 < |z-1| < \infty)$$

this is *the* Laurent series for f in D_5 where $a_n = 0$, $b_1 = 0, b_n = -1, n \geq 2$

- **domain** D_6**:** $0 < |z-2| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{(1+z-2)} - \frac{1}{z-2}$$
$$= -\frac{1}{z-2} + \sum_{n=0}^{\infty} (-1)^n (z-2)^n \qquad (0 < |z-2| < 1)$$

this is *the* Laurent series for f in D_4 with $b_1 = -1, b_n = 0, n \geq 2$, $a_n = (-1)^n$

- **domain** D_7: $1 < |z-2|$ $\quad$ ($1/|z-2| < 1$ for all $z \in D_7$)

$$f(z) = \frac{-1}{(z-2+1)(z-2)} = \frac{-1}{(z-2)^2} \cdot \frac{1}{1+1/(z-2)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(z+2)^{n+2}}, \qquad (1 < |z-2| < \infty)$$

the Laurent series for f in D_7 where $a_n = 0$, $b_1 = 0, b_n = (-1)^{n+1}, n \geq 2$

remark: we can find related integrals from the coefficients of the Laurent series

for example, let C be a simple positive closed contour lying in D_7

$$\begin{aligned}
\int_C \tfrac{-1}{(z-1)(z-2)}\, dz &= \int_C f(z)dz &= j2\pi b_1 &= 0 \\
\int_C \tfrac{-1}{(z-1)(z-2)^2}\, dz &= \int_C \tfrac{f(z)}{(z-2)}dz &= j2\pi a_0 &= 0 \\
\int_C \tfrac{-1}{(z-1)}\, dz &= \int_C f(z)(z-2)dz &= j2\pi b_2 &= -j2\pi
\end{aligned}$$

example: find a Laurent series for $f(z) = \dfrac{e^z}{(z+1)^2}$ in a certain domain

for any z, since e^z has a Maclaurin series about 0, we can write

$$\begin{aligned}
\frac{e^z}{(z+1)^2} &= \frac{e^{z+1}}{e(z+1)^2} = \frac{1}{e}\sum_{n=0}^{\infty} \frac{(z+1)^n}{n!(z+1)^2} \\
&= \frac{1}{e}\sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!} \\
&= \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right], \qquad (0 < |z+1| < \infty)
\end{aligned}$$

this is the Laurent series for f in the domain $0 < |z+1| < \infty$ where

$$b_1 = 1/e, \quad b_2 = 1/e, \quad b_k = 0, \forall k \geq 3, \quad a_n = \frac{1/e}{(n+2)!}$$

References

Chapter 5 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

Chapter 22 in

M. Dejnakarin, *Mathematics for Electrical Engineering*, CU Press, 2006