

## 3. Vectors and Matrices

- review on vectors
- matrix notation
- special matrices
- matrix operations
- inverse of matrices
- elementary matrices
- determinants
- linear equations in matrix form

# Vector notation

$n$ -vector  $x$ :

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- also written as  $x = (x_1, x_2, \dots, x_n)$
- set of  $n$ -vectors is denoted  $\mathbf{R}^n$  (Euclidean space)
- $x_i$ :  $i$ th **element** or **component** or **entry** of  $x$
- $x$  is also called a column vector
- $y = [y_1 \quad y_2 \quad \cdots \quad y_n]$  is called a row vector

unless stated otherwise, a vector typically means a column vector

## Special vectors

**zero vectors:**  $x = (0, 0, \dots, 0)$

**all-ones vectors:**  $x = (1, 1, \dots, 1)$  (we will denote it by **1**)

**standard unit vectors:**  $e_k$  has only 1 at the  $k$ th entry and zero otherwise

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(standard unit vectors in  $\mathbf{R}^3$ )

**unit vectors:** any vector  $u$  whose norm (magnitude) is 1, *i.e.*,

$$\|u\| \triangleq \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = 1$$

example:  $u = (1/\sqrt{2}, 2/\sqrt{6}, -1/\sqrt{2})$

# Vector operations

**scalar multiplication** of a vector  $x$  with a scalar  $\alpha$

$$\alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

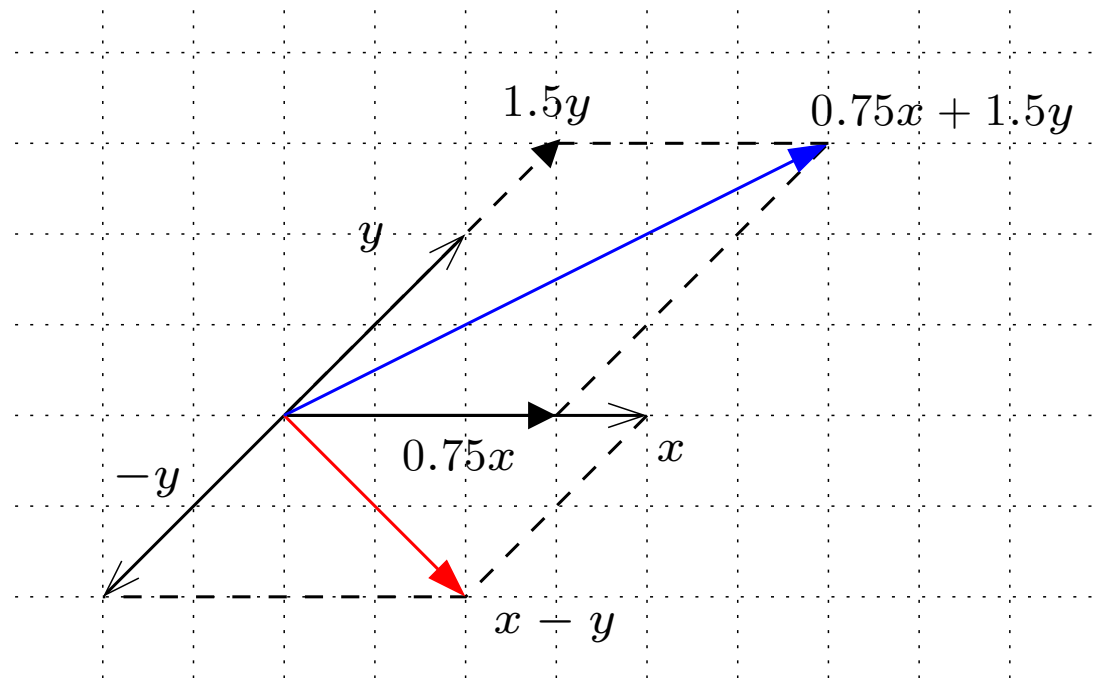
**addition and subtraction** of two  $n$ -vector  $x, y$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad x - y = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{bmatrix}$$

# Geometrical interpretation

for  $n \leq 3$ :  $x$  is a point with coordinates  $x_i$

**example:**  $x = (4, 0)$ ,  $y = (2, 2)$



$$0.75x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \quad 1.5y = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad 0.75x + 1.5y = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad x - y = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

# Inner products

**definition:** the inner product of two  $n$ -vectors  $x, y$  is

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

also known as the **dot product** of vectors  $x, y$

**notation:**  $x^T y$

**properties** 

- $(\alpha x)^T y = \alpha(x^T y)$  for scalar  $\alpha$
- $(x + y)^T z = x^T z + y^T z$
- $x^T y = y^T x$

# Euclidean norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

## properties

- also written  $\|x\|_2$  to distinguish from other norms
- $\|\alpha x\| = |\alpha| \|x\|$  for scalar  $\alpha$
- $\|x + y\| \leq \|x\| + \|y\|$  (triangle inequality)
- $\|x\| \geq 0$  and  $\|x\| = 0$  only if  $x = 0$

## interpretation

- $\|x\|$  measures the *magnitude* or length of  $x$
- $\|x - y\|$  measures the *distance* between  $x$  and  $y$

# Matrix notation

an  $m \times n$  matrix  $A$  is defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \text{ or } A = [a_{ij}]_{m \times n}$$

- $a_{ij}$  are the **elements**, or **coefficients**, or **entries** of  $A$
- set of  $m \times n$ -matrices is denoted  $\mathbf{R}^{m \times n}$
- $A$  has  $m$  rows and  $n$  columns ( $m, n$  are the **dimensions**)
- the  $(i, j)$  entry of  $A$  is also commonly denoted by  $A_{ij}$
- $A$  is called a **square** matrix if  $m = n$



## Special matrices

**zero matrix:**  $A = 0$

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$a_{ij} = 0$ , for  $i = 1, \dots, m, j = 1, \dots, n$

**identity matrix:**  $A = I$

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

a square matrix with  $a_{ii} = 1, a_{ij} = 0$  for  $i \neq j$

**diagonal matrix:**

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{bmatrix}$$

a square matrix with  $a_{ij} = 0$  for  $i \neq j$

**triangular matrix:**

a square matrix with zero entries in a triangular part

**upper triangular**

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$a_{ij} = 0 \text{ for } i \geq j$$

**lower triangular**

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$a_{ij} = 0 \text{ for } i \leq j$$

# Addition and scalar multiplication

**addition** of two  $m \times n$ -matrices  $A$  and  $B$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

**scalar multiplication** of an  $m \times n$ -matrix  $A$  with a scalar  $\beta$

$$\beta A = \begin{bmatrix} \beta a_{11} & \beta a_{12} & \dots & \beta a_{1n} \\ \beta a_{21} & \beta a_{22} & \dots & \beta a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta a_{m1} & \beta a_{m2} & \dots & \beta a_{mn} \end{bmatrix}$$

# Multiplication

product of  $m \times r$ -matrix  $A$  with  $r \times n$ -matrix  $B$ :

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj} = \sum_{k=1}^r a_{ik} b_{kj}$$

dimensions must be compatible: # of columns in  $A = \#$  of rows in  $B$

- $(AB)_{ij}$  is the dot product of the  $i^{\text{th}}$  row of  $A$  and the  $j^{\text{th}}$  column of  $B$
- $AB \neq BA$  in general ! (even if the dimensions make sense)
- there are exceptions, *e.g.*,  $AI = IA$  for all square  $A$
- $A(B + C) = AB + AC$

# Matrix transpose

the transpose of an  $m \times n$ -matrix  $A$  is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

properties 

- $A^T$  is  $n \times m$
- $(A^T)^T = A$
- $(\alpha A + B)^T = \alpha A^T + B^T$ ,  $\alpha \in \mathbf{R}$
- $(AB)^T = B^T A^T$
- a square matrix  $A$  is called **symmetric** if  $A = A^T$ , *i.e.*,  $a_{ij} = a_{ji}$

# Block matrix notation

**example:**  $2 \times 2$ -block matrix  $A$

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

for example, if  $B, C, D, E$  are defined as

$$B = \begin{bmatrix} 2 & 1 \\ 3 & 8 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 7 \\ 1 & 9 & 1 \end{bmatrix}, \quad D = [0 \quad 1], \quad E = [-4 \quad 1 \quad -1]$$

then  $A$  is the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 1 & 7 \\ 3 & 8 & 1 & 9 & 1 \\ 0 & 1 & -4 & 1 & -1 \end{bmatrix}$$

note: dimensions of the blocks must be compatible

# Column and Row partitions

write an  $m \times n$ -matrix  $A$  in terms of its columns or its rows

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n] = \begin{bmatrix} b_1^T \\ b_2^T \\ \vdots \\ b_m^T \end{bmatrix}$$

- $a_j$  for  $j = 1, 2, \dots, n$  are the columns of  $A$
- $b_i^T$  for  $i = 1, 2, \dots, m$  are the rows of  $A$

**example:**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 9 & 0 \end{bmatrix}$$

the column and row vectors are

$$a_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_1^T = [1 \quad 2 \quad 1], \quad b_2^T = [4 \quad 9 \quad 0]$$

# Matrix-vector product

product of  $m \times n$ -matrix  $A$  with  $n$ -vector  $x$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

- dimensions must be compatible:  $\#$  columns in  $A = \#$  elements in  $x$

if  $A$  is partitioned as  $A = [a_1 \ a_2 \ \dots \ a_n]$ , then

$$Ax = a_1x_1 + a_2x_2 + \dots + a_nx_n$$

- $Ax$  is a linear combination of the column vectors of  $A$
- the coefficients are the entries of  $x$



## Product with standard unit vectors

post-multiply with a column vector

$$Ae_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{mk} \end{bmatrix} = \text{the } k\text{th column of } A$$

pre-multiply with a row vector

$$e_k^T A = [0 \ 0 \ \cdots \ 1 \ \cdots \ 0] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ = [a_{k1} \ a_{k2} \ \cdots \ a_{kn}] = \text{the } k\text{th row of } A$$

# Trace

## Definition:

trace of a square matrix  $A$  is the sum of the diagonal entries in  $A$

$$\mathbf{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

## example:

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & -1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

trace of  $A$  is  $2 - 1 + 6 = 7$

## properties ↗

- $\mathbf{tr}(A^T) = \mathbf{tr}(A)$
- $\mathbf{tr}(\alpha A + B) = \alpha \mathbf{tr}(A) + \mathbf{tr}(B)$
- $\mathbf{tr}(AB) = \mathbf{tr}(BA)$

# Inverse of matrices

## Definition:

a *square* matrix  $A$  is called **invertible** or **nonsingular** if there exists  $B$  s.t.

$$AB = BA = I$$

- $B$  is called an **inverse** of  $A$
- it is also true that  $B$  is invertible and  $A$  is an inverse of  $B$
- if no such  $B$  can be found  $A$  is said to be **singular**

assume  $A$  is invertible

- an inverse of  $A$  is unique
- the inverse of  $A$  is denoted by  $A^{-1}$

assume  $A, B$  are invertible

## Facts

- $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$  for nonzero  $\alpha$
- $A^T$  is also invertible and  $(A^T)^{-1} = (A^{-1})^T$
- $AB$  is invertible and  $(AB)^{-1} = B^{-1}A^{-1}$
- $(A + B)^{-1} \neq A^{-1} + B^{-1}$

## Inverse of $2 \times 2$ matrices

the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if

$$ad - bc \neq 0$$

and its inverse is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**example:**

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}, \quad A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

# Elementary matrices

**Definition:** a matrix obtained by performing a *single* row operation on the identity matrix  $I_n$  is called an **elementary** matrix

**examples:**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$$

add  $k$  times the first row to the third row of  $I_3$

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

multiply a nonzero  $k$  with the second row of  $I_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

interchange the second and the third rows of  $I_3$

an elementary matrix is often denoted by  $E$

# Inverse operations

row operations on  $E$  that produces  $I$  and vice versa

$I \rightarrow E$	$E \rightarrow I$
add $k$ times row $i$ to row $j$	add $-k$ times row $i$ to row $j$
multiply row $i$ by $k \neq 0$	multiply row $i$ by $1/k$
interchange row $i$ and $j$	interchange row $i$ and $j$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -k & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Facts

- every elementary matrix is invertible
- the inverse is also an elementary matrix

from the examples in page 3-23

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -k & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/k \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \implies E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



## Row operations by matrix multiplication

assume  $A$  is  $m \times n$  and  $E$  is obtained by performing a row operation on  $I_m$

$EA$  = the matrix obtained by performing this same row operation on  $A$

**example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

- add  $-2$  times the third row to the second row of  $A$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad EA = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

- multiply 2 with the first row of  $A$

$$E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad EA = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

- interchange the first and the third rows of  $A$

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad EA = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

# Inverse via row operations

assume  $A$  is invertible

- $A$  is reduced to  $I$  by a finite sequence of row operations

$$E_1, E_2, \dots, E_k$$

such that

$$E_k \cdots E_2 E_1 A = I$$

- the reduced echelon form of  $A$  is  $I$
- the inverse of  $A$  is therefore given by the product of elementary matrices

$$A^{-1} = E_k \cdots E_2 E_1$$

**example:** write the augmented matrix  $[A \mid I]$

$$\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}$$

and apply row operations until the left side is reduced to  $I$

$$\begin{array}{l} -2R_2 + R_1 \rightarrow R_1 \\ -R_2 + R_3 \rightarrow R_3 \\ \\ R_1 \leftrightarrow R_2 \\ \\ -3R_2 + R_3 \rightarrow R_3 \end{array} \quad \begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & -2 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & -1 & 1 \\ \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 3 & 0 & -1 & 1 \\ \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & -2 & 0 & -3 & 5 & 1 \end{array}$$

$$\begin{array}{l}
 R_3/(-2) \rightarrow R_3 \\
 \\
 R_2 \leftrightarrow R_3 \\
 \\
 -2R_2 + R_1 \rightarrow R_1 \\
 \\
 -R_3 + R_1 \rightarrow R_1
 \end{array}
 \begin{array}{l}
 \begin{array}{ccc|ccc}
 1 & 2 & 1 & 0 & 1 & 0 \\
 0 & 0 & 1 & 1 & -2 & 0 \\
 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2}
 \end{array} \\
 \\
 \begin{array}{ccc|ccc}
 1 & 2 & 1 & 0 & 1 & 0 \\
 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\
 0 & 0 & 1 & 1 & -2 & 0
 \end{array} \\
 \\
 \begin{array}{ccc|ccc}
 1 & 0 & 1 & -3 & 6 & 1 \\
 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\
 0 & 0 & 1 & 1 & -2 & 0
 \end{array} \\
 \\
 \begin{array}{ccc|ccc}
 1 & 0 & 0 & -4 & 8 & 1 \\
 0 & 1 & 0 & \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\
 0 & 0 & 1 & 1 & -2 & 0
 \end{array}
 \end{array}$$

the inverse of  $A$  is

$$\begin{bmatrix}
 -4 & 8 & 1 \\
 \frac{3}{2} & -\frac{5}{2} & -\frac{1}{2} \\
 1 & -2 & 0
 \end{bmatrix}$$

# Invertible matrices

✌ **Theorem:** for a square matrix  $A$ , the following statements are equivalent

1.  $A$  is invertible
2.  $Ax = 0$  has only the trivial solution ( $x = 0$ )
3. the reduced echelon form of  $A$  is  $I$
4.  $A$  is expressible as a product of elementary matrices

## Inverse of special matrices

diagonal matrix

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n \end{bmatrix}$$

a diagonal matrix is invertible iff the diagonal entries are all nonzero

$$a_{ii} \neq 0, \quad i = 1, 2, \dots, n$$

the inverse of  $A$  is given by

$$A^{-1} = \begin{bmatrix} 1/a_1 & 0 & \cdots & 0 \\ 0 & 1/a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1/a_n \end{bmatrix}$$

the diagonal entries in  $A^{-1}$  are the inverse of the diagonal entries in  $A$

## triangular matrix:

upper triangular

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$a_{ij} = 0 \text{ for } i \geq j$$

lower triangular

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$a_{ij} = 0 \text{ for } i \leq j$$

a triangular matrix is invertible iff the diagonal entries are all nonzero

$$a_{ii} \neq 0, \quad \forall i = 1, 2, \dots, n$$

more is true ...

- product of lower (upper) triangular matrices is lower (upper) triangular
- the inverse of a lower (upper) triangular matrix is lower (upper) triangular



**symmetric matrix:**  $A = A^T$



- for any square matrix  $A$ ,  $AA^T$  and  $A^T A$  are always symmetric
- if  $A$  is symmetric and invertible, then  $A^{-1}$  is symmetric
- if  $A$  is invertible, then  $AA^T$  and  $A^T A$  are also invertible

# Determinants

the determinant is a *scalar value* associated with a square matrix  $A$

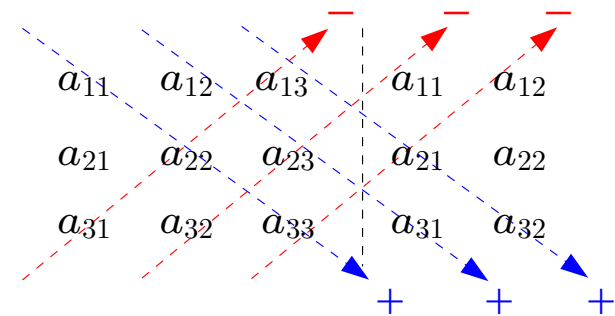
commonly denoted by  $\det(A)$  or  $|A|$

determinants of  $2 \times 2$  matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

determinants of  $3 \times 3$  matrices: let  $A = \{a_{ij}\}$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$



for a square matrix of any order, it can be computed by

- cofactor expansion
- performing elementary row operations

# Minor and Cofactor

**Minor of entry  $a_{ij}$ :** denoted by  $M_{ij}$

- the determinant of the resulting submatrix after deleting the  $i$ th row and  $j$ th column of  $A$

**Cofactor of entry  $a_{ij}$ :** denoted by  $C_{ij}$

- $C_{ij} = (-1)^{(i+j)} M_{ij}$

**example:**

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad M_{23} = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4, \quad C_{23} = (-1)^{(2+3)} M_{23} = 4$$

# Determinants by Cofactor Expansion

**Theorem:** the determinant of an  $n \times n$ -matrix  $A$  is given by

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

regardless of which row or column of  $A$  is chosen

**example:** pick the first row to compute  $\det(A)$

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{aligned} \det(A) &= 3(-1)^2 \begin{vmatrix} 0 & 2 \\ -1 & 2 \end{vmatrix} + 1(-1)^3 \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} - 2(-1)^4 \begin{vmatrix} 5 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 3(1)(2) + (-1)(8) - 2(1)(-5) = 8 \end{aligned}$$

# Basic properties of determinants

✌ let  $A, B$  be any square matrices

- $\det(A) = \det(A^T)$
- if  $A$  has a row of zeros or a column of zeros, then  $\det(A) = 0$
- $\det(A + B) \neq \det(A) + \det(B)$  !

determinants of special matrices:

- the determinant of a diagonal or triangular matrix is given by the product of the diagonal entries
- $\det(I) = 1$

(these properties can be proved from the def. of cofactor expansion)

✌ another basic properties; suppose the following is true

- $A$  and  $B$  are equal except for the entries in their  $k$ th row (column)
- $C$  is defined as that matrix identical to  $A$  and  $B$  except that its  $k$ th row (column) is the sum of the  $k$ th rows (columns) of  $A$  and  $B$

then we have

$$\det(C) = \det(A) + \det(B)$$

example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\det(A) = 0, \quad \det(B) = -1, \quad \det(C) = -1$$

# Determinants under row operations

- multiply  $k$  to a row or a column

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- interchange between two rows or two columns

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- add  $k$  times the  $i$ th row (column) to the  $j$ th row (column)

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



(the proof of determinants under row operations is left as an exercise )

example:  $B$  is obtained by performing the following operations on  $A$

$$R_2 + 3R_1 \rightarrow R_2, \quad R_3 \leftrightarrow R_1, \quad -4R_1 \rightarrow R_1$$

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 3 & 1 & 0 \\ -3 & -3 & 3 \end{bmatrix} \implies \det(B) = (-4) \cdot (-1) \cdot 1 \cdot \det(A)$$

the changes of det. under elementary operations lead to obvious facts 

- $\det(\alpha A) = \alpha^n \det(A), \quad \alpha \neq 0$
- If  $A$  has two rows (columns) that are equal, then  $\det(A) = 0$

## Determinants of elementary matrices

let  $B$  be obtained by performing a row operation on  $A$  then

$$B = EA \quad \text{and} \quad \det(B) = \det(EA)$$

$$E = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = k \det(A) \quad (\det(E) = k)$$

$$E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = -\det(A) \quad (\det(E) = -1)$$

$$E = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \det(B) = \det(A) \quad (\det(E) = 1)$$

**conclusion:**  $\det(EA) = \det(E) \det(A)$

# Determinants of product and inverse

✌ let  $A, B$  be  $n \times n$  matrices

- $A$  is invertible if and only if  $\det(A) \neq 0$
- if  $A$  is invertible, then  $\det(A^{-1}) = 1/\det(A)$
- $\det(AB) = \det(A)\det(B)$

# Adjoint

the adjoint of  $A$  is the transpose of the matrix of cofactors from  $A$

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

if  $A$  is invertible then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

**Proof.**

- the cofactor expansion using the cofactors from different row is zero

$$a_{i1}C_{k1} + a_{i2}C_{k2} + \cdots + a_{in}C_{kn} = 0, \quad \text{for } i \neq k$$

- $A \text{adj}(A) = \det(A) \cdot I$

# Linear equation in matrix form

the linear system of  $m$  equations in  $n$  variables

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots = \vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

in matrix form:  $Ax = b$  where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

# Applications

a set of linear equations  $Ax = b$  (with  $A$   $m \times n$ ) is

- **square** if  $m = n$

( $A$  is square)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **underdetermined** if  $m < n$

( $A$  is fat)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- **overdetermined** if  $m > n$

( $A$  is skinny)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

## Cramer's rule

consider a linear system  $Ax = b$  when  $A$  is **square**

if  $A$  is invertible then the solution is unique and given by

$$x = A^{-1}b$$

each component of  $x$  can be calculated by using the Cramer's rule

### Cramer's rule

$$x_1 = \frac{|A_1|}{|A|}, \quad x_2 = \frac{|A_2|}{|A|}, \quad \dots, \quad x_n = \frac{|A_n|}{|A|}$$

where  $A_j$  is the matrix obtained by replacing  $b$  in the  $j$ th column of  $A$

(its proof is left as an exercise)

**example:**

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 5 & 0 & 2 \\ 1 & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

since  $\det(A) = 8$ ,  $A$  is invertible and the solution is

$$x = A^{-1}b = \frac{1}{8} \begin{bmatrix} 2 & 0 & 2 \\ -8 & 8 & -16 \\ -5 & 4 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}$$

using Cramer's rule gives

$$x_1 = \frac{1}{8} \begin{vmatrix} 2 & 1 & -2 \\ 1 & 0 & 2 \\ 2 & -1 & 2 \end{vmatrix}, \quad x_2 = \frac{1}{8} \begin{vmatrix} 3 & 2 & -2 \\ 5 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix}, \quad x_3 = \frac{1}{8} \begin{vmatrix} 3 & 1 & 2 \\ 5 & 0 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

which yields

$$x_1 = 1, \quad x_2 = -5, \quad x_3 = -2$$



# MATLAB commands

some commonly used commands for working with matrices

- `eye(n)` produces an identity matrix of size  $n$
- `zeros(m,n)` creates a zero matrix of size  $m \times n$
- `inv(A)` finds the inverse of  $A$
- `det(A)` finds the determinant of  $A$
- `trace(A)` finds the trace of  $A$

to solve  $Ax = b$  when  $A$  is square use

- `A\b` (compute  $A^{-1}b$ )

# References

Chapter 1-2 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010

Lecture note on

*Matrices and Vectors*, EE103, L. Vandenberghe, UCLA