1. Mathematical Proofs

- conditional statements
- sufficient and necessary conditions
- methods of proofs
- disproving statements
- proofs of quantified statements

Statements

a **statement** is a declarative sentence that is true *or* false but not both

examples:

- $3 + 4 = 7$
- $5 \cdot 2 - 3 = 9$
- if x is an integer, then $2x$ is an even integer

the following sentences are not statements

- Bangkok is a lovely city (it's a matter of opinion)
- $2x - 3 = 4$ (we do not know what x is)

Conditional statements

for statements P and Q, a **conditional statement** is the statement:

If P, **then** Q

and is denoted by $P \Rightarrow Q$ (also stated as P implies Q)

example: 'if students obtain a score higher than 80 then they will get an A'

truth table

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \Rightarrow Q$ is logically equivalent to

- $\neg P \vee Q$
- $\neg Q \Rightarrow \neg P$

beware ! $P \Rightarrow Q$ is NOT logically equivalent to $Q \Rightarrow P$

Biconditional statements

the conjunction of a conditional statement and its *converse*:

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

is called the **biconditional** of P and Q, which is expressed as

$$P \textbf{ if and only if } Q$$

and denoted by $P \Leftrightarrow Q$

truth table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

examples:

- $x = 2$ if and only if $3x = 6$
- $|x| = 4$ if and only if $x^2 = 16$

$P \Leftrightarrow Q$ is true only when P and Q have the same truth values

Sufficient and Necessary conditions

consider a (true) conditional statement: $P \Rightarrow Q$, we say

- P is **sufficient** for Q
- Q is **necessary** for P
- P **only if** Q

example: if $x = -3$ then $|x| = 3$ (a true conditional statement)

- 'P is sufficient for Q' means

  the truth of $x = -3$ is sufficient for concluding the truth of $|x| = 3$

- 'P only if Q' and 'Q is necessary for P' have the same meaning:

  $x = -3$ is *true only* under the condition that $|x| = 3$ (because if $|x| \neq 3$ then $x = -3$ can't be true) however, $|x| = 3$ is *not a sufficient condition* for $x = -3$

(because if $|x| = 3$ then x can be either 3 or -3)

$i.e.$, the converse of 'if $x = -3$ then $|x| = 3$' is false

consider a (true) biconditional statement: $P \Leftrightarrow Q$, we say

P is **sufficient** and **necessary** for Q

example: $|x| = 2$ if and only if $x^2 = 4$ (a true biconditional statement)

- saying $|x| = 2$ is equivalent to saying $x^2 = 4$

more examples:

- being at least 18 years old is *necessary* for applying a driver license

  $i.e.,$

  - if you're a driver, everyone knows you must be at least 18 years old
  - if you're younger than 18 then you can't have a driver license

- if a person holds the title 'Miss Thailand' then that person must be 1) female 2) adult and 3) unmarried

  $i.e.,$

  - stating that 'Jenny is Miss Thailand' is *sufficient* to know that she is female and she must be old enough (an adult)
  - being unmarried is a *necessary* condition for being Miss Thailand because if a woman is married, she can't apply for this position

Mathematical terminology

- an **axiom** is a math statement that is *self-evidently true* w/o proof
- a **definition** is an *agreement* as to the meaning of a particular term
- a **proof** is a sequence of math arguments demonstrating the truth of given results
- a **theorem** or a **proposition** is any mathematical statement that can be shown to be true using accepted logical and mathematical arguments
- a **lemma** is a true mathematical statement that was proven mainly to help in the proof of some theorem
- a **corollary** is used to refer to a theorem that is easily proven once some other theorem has been proven

Direct proofs

a **direct proof** of $P \Rightarrow Q$ typically consists of these steps:

1. start from assuming P is true then
2. develop a set of logical arguments to conclude Q

example: show that if $x, y \in \mathbf{R}$ then $x^2 + y^2 \geq |xy|$

Proof. let $x, y \in \mathbf{R}$ and consider $(|x| - |y|)^2$

$$(|x| - |y|)^2 = |x|^2 + |y|^2 - 2|xy|$$

since the LHS is nonnegative, it follows that

$$(|x| - |y|)^2 = x^2 + y^2 - 2|xy| \geq 0$$

and hence $x^2 + y^2 \geq 2|xy| \geq |xy|$ $\square$

Proof by contrapositive

a **contrapositive proof** of a statement $P \Rightarrow Q$ uses the fact that

$$P \Rightarrow Q \quad \text{is logically equivalent to} \quad \neg Q \Rightarrow \neg P$$

so we can use a direct proof to show that $\neg Q \Rightarrow \neg P$ is true

example: let $x \in \mathbf{R}$. show that if $x^2 + 2x < 0$ then $x < 0$

Proof. we will show that if $x \geq 0$ then $x^2 + 2x \geq 0$

- if $x \geq 0$ then obviously $2x \geq 0$
- x^2 is always nonnegative

therefore, the sum of x^2 and $2x$ is nonnegative, finishing the proof $\square$

Proof by contradiction

idea: $\neg(P \Rightarrow Q)$ is equivalent to $P \wedge \neg Q$, so if we do as follows:

1. assume P is true (accept all the hypotheses) and Q is false (negate the conclusion)
2. try to prove that this leads to a **contradiction**

then we have shown that $\neg(P \Rightarrow Q)$ is false or that $P \Rightarrow Q$ is true

example: show that if n is an even integer then so is n^2

Proof. assume n is even but n^2 is not

since n is even, we can express $n = 2k$ where k is some positive integer

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

since $2k^2$ is also an integer, n^2 must be also even, which is a contradiction

Proof by induction

principle of mathematical indunction states that

the statement $P(n)$ is true for all $n \in \mathrm{N}$ if

1. $P(1)$ is true

2. for each $k \in \mathrm{N}$, if $P(k)$ is true then $P(k+1)$ is also true

example: show that $\sum_{i=1}^{n} i = n(n+1)/2$ for $n = 1, 2, \ldots$

Proof. let $P(n)$ be the statement $\sum_{i=1}^{n} i = n(n+1)/2$

- $P(1)$ is true because $1 = 1 \cdot (1+1)/2$
- assume $P(k)$ is true and show that $P(k+1)$ is true:

$$\sum_{i=1}^{n+1} i = n + 1 + \sum_{i=1}^{n} i = n + 1 + n(n+1)/2 = (n+1)(n+2)/2$$

Disproving statements

a **conjecture** is any math statement that has *not* been proved or disproved

disproving a conjecture requires only a *single example* to show the conjecture is *false*

such example is called a **counterexample**

example: $(x + y)^2 = x^2 + y^2$ for all $x, y \in \mathbf{R}$ (conjecture)

$x = 1, y = 1$ is a counterexample that disproves the conjecture because

$$(1 + 1)^2 = 4 \neq 1^2 + 1^2 = 2$$

(because the conjecture says the identity holds *for all* x, y, we just gave a value of x, y that disproves it)

example: let A be a square matrix. if $A^2 = I$ then $A = I$ or $-I$

the conjecture is false because if we consider

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then we can verify that

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

hence, $A^2 = I$ does not necessarily imply that $A = I$ or $A = -I$

but A could be other matrices (at least the counterexample we just gave)

Quantifiers

- the quantifying clause '**for every, for all, for each**' is denoted by $\forall$
- the quantifying clause '**there exists, there is some**' is denoted by $\exists$
- $x \in S$ means 'x is a member of set S' or 'x belongs to S'

examples:

- for every positive real number x, $x^3 - 2x^2 + x > 0$

$$\forall x \in \mathbf{R}, \;\; x^3 - 2x^2 + x > 0$$

- there exists a real number x such that $x^2 - 2x = 4$

$$\exists x, \;\; x^2 - 2x = 4$$

Proofs of quantified statements

statements containing 'for some' or 'there exists'

example: prove or disprove '$\exists A \in \mathbf{R}^{2\times 2},\ \det(A) = 1$'

to prove that it's true, we just need to come up with *an example* of A:

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and show that} \quad \det(A) = 1$$

hence, the statement is true

example: prove or disprove '$\exists x \in \mathbf{R},\ x^4 + 2x^2 + 1 = 0$'

if $x \in \mathbf{R}$, then $x^4 \geq 0$ and $x^2 \geq 0$, so $x^4 + 2x^2 + 1 \geq 1$

$x^4 + 2x^2 + 1$ can't be 0 **for any** $x \in \mathbf{R}$, so the statement is false

- proving that the statement is true is typically (but not always) *simple*
- disproving the statement may require some effort

statements containing 'for all' or 'for any'

example: prove or disprove '$\forall x, y \in \mathbf{R},\ |x+y| \le |x| + |y|$'

$$(x+y)^2 = x^2 + y^2 + 2xy \le |x|^2 + |y|^2 + 2|xy| = (|x| + |y|)^2$$

so the statement is true

example: prove or disprove '$AB = BA$ for any square matrices A, B'

disproving it is easy because we can just give an example of A, B:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and show that $AB = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (so the statement is false)

- proving the statement is true may require some effort
- disproving the statement is typically *easy* (by giving a counterexample)

Common mistakes

example: show that for any $\alpha \in \mathbf{R}, A \in \mathbf{R}^{n\times n}$, $\det(\alpha A) = |\alpha|^n \det A$

one may show as follows

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \Longrightarrow \quad \det(A) = 5 \text{ and } \det(\alpha A) = \begin{vmatrix} \alpha & 2\alpha \\ -\alpha & 3\alpha \end{vmatrix} = 5\alpha^2$$

so $\det(\alpha A) = \alpha^2 \det(A)$ as desired

the above argument *cannot* be a proof because we just showed for *one* particular value of A

in fact, we have to show that the statement is true **for all** square matrices

example: show that for any $x, y \in \mathbf{R}$, $\quad (x+y)^2 \leq 2(x^2+y^2)$

if one writes an argument like this:

$$x^2 + 2xy + y^2 \leq 2x^2 + 2y^2 \;\Rightarrow\; x^2 + y^2 - 2xy \geq 0 \;\Rightarrow\; (x-y)^2 \geq 0$$

then it can't be a proof because:

- we can't start a proof from the result we're going to prove !
- each step of argument must be explained with logical reasoning
- a good proof must be clear by itself; always explain with details
- the lastly obtained result must conclude what you want to prove

example of proof: for any $x, y \in \mathbf{R}$, $(x-y)^2$ is always nonnegative

- expanding $(x-y)^2$ gives

$$0 \leq (x-y)^2 = x^2 - 2xy + y^2$$

- add $x^2 + 2xy + y^2$ on both sides

$$x^2 + 2xy + y^2 \leq 2x^2 + 2y^2$$

- complete the square and we finish the proof

$$(x+y)^2 \leq 2(x^2 + y^2)$$

References

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