2. System of Linear Equations

- linear equations
- elementary row operations
- Gaussian elimination

Linear equations

a general linear system of m equations with n variables is described by

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & = & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

where a_{ij}, b_j are constants and $x_1, x_2, \ldots, x_n$ are unknowns

- equations are linear in $x_1, x_2, \ldots, x_n$
- existence and uniqueness of a solution depend on a_{ij} and b_j

Example: solving ordinary differential equations

given $y(0) = 1, \dot{y}(0) = -1, \ddot{y}(0) = 0$, solve

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 0$$

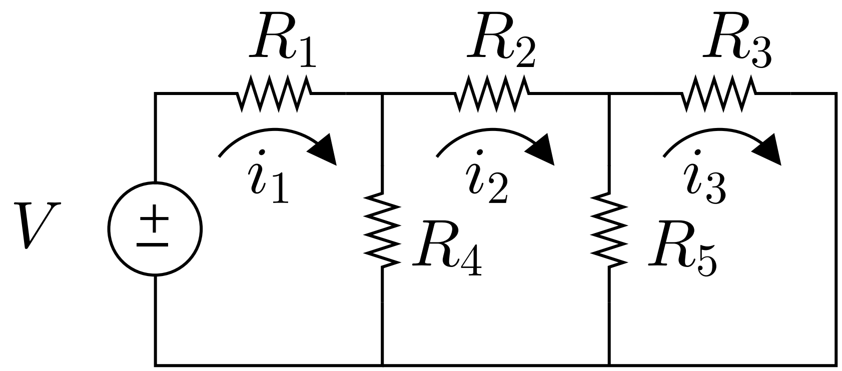
the closed-form solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{-3t}$$

C_1, C_2 and C_3 can be found by solving a set of linear equations

$$\begin{array}{rcccl} 1 & = & y(0) & = & C_1 + C_2 + C_3 \\ -1 & = & \dot{y}(0) & = & -C_1 - 2C_2 - 3C_3 \\ 0 & = & \ddot{y}(0) & = & C_1 + 4C_2 + 9C_3 \end{array}$$

Example: linear static circuit



given V, $R_1, R_2, \ldots, R_5$, find the currents in each loop

by KVL, we obtain a set of linear equations

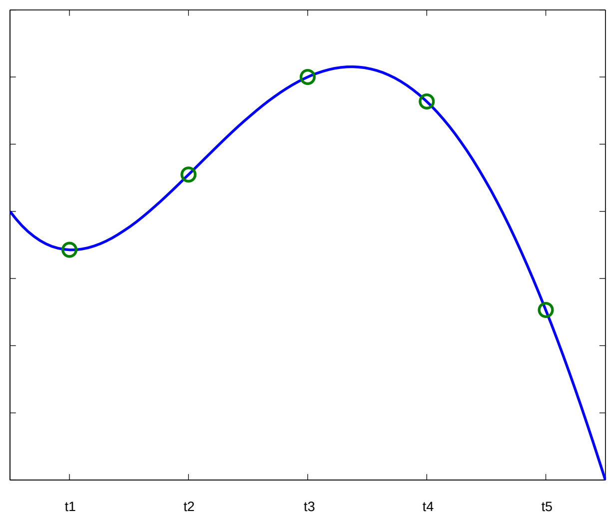
$$\begin{aligned} V &= (R_1 + R_4)i_1 - R_4 i_2 \\ 0 &= -R_4 i_1 + (R_2 + R_4 + R_5)i_2 - R_5 i_3 \\ 0 &= -R_5 i_2 + (R_3 + R_5)i_3 \end{aligned}$$

Example: polynomial interpolation

fit a polynomial

$$p(t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1}$$

through n points $(t_1, y_1), \ldots, (t_n, y_n)$



problem data (parameters): $t_1, \ldots, t_n, y_1, \ldots, y_n$

problem variables: find $x_1, \ldots, x_n$ such that $p(t_i) = y_i$ for all i

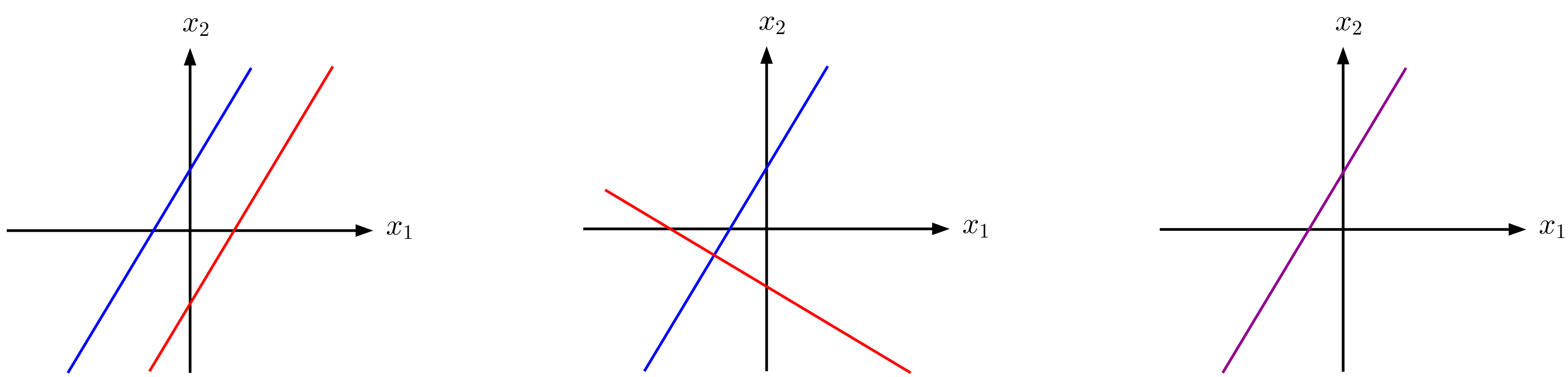
write out the conditions on x:

$$\begin{array}{rcl}
p(t_1) & = & x_1 + x_2 t_1 + x_3 t_1^2 + \cdots + x_n t_1^{n-1} & = & y_1 \\
p(t_2) & = & x_1 + x_2 t_2 + x_3 t_2^2 + \cdots + x_n t_2^{n-1} & = & y_2 \\
\vdots & & \vdots & & \vdots \\
p(t_n) & = & x_1 + x_2 t_n + x_3 t_n^2 + \cdots + x_n t_n^{n-1} & = & y_n
\end{array}$$

Special case: two variables

Examples:

$$\begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= -2 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ x_1 + x_2 &= -1 \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= -1 \\ 4x_1 - 2x_2 &= -2 \end{aligned}$$



(a) no solution (b) one solution (c) infinitely many solutions

- no solution if two lines are parallel but different interceptions on x_2-axis
- many solutions if the two lines are identical

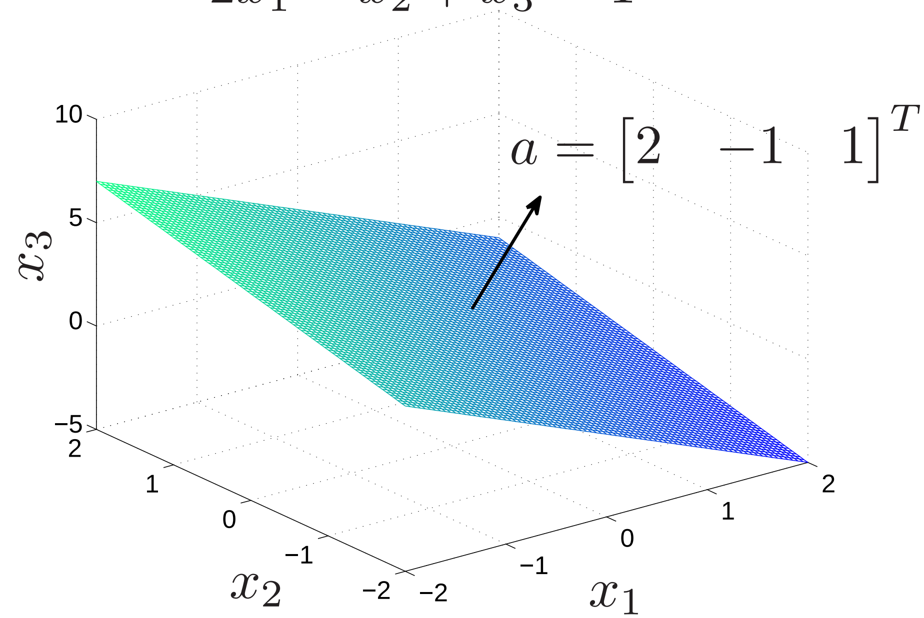
Geometrical interpretation

the set of solutions to a linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

can be interpreted as a hyperplane on $\mathbf{R}^n$

$$2x_1 - x_2 + x_3 = 1$$



a solution to m linear equations is an **intersection** of m hyperplanes

Existence and uniqueness of solutions

existence:

- no solution
- a solution exists

  uniqueness:

  - the solution is unique
  - there are infinitely many solutions

every system of linear equations has zero, one, or infinitely many solutions

there are no other possibilities

no solution

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= 2 \end{aligned}$$

unique solution

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 - x_2 &= 0 \end{aligned} \qquad \begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= -1 \\ x_1 - x_2 &= -2 \end{aligned}$$

infinitely many solutions

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ -x_1 + x_3 &= -1 \\ 3x_1 - 2x_2 + 3x_3 &= 3 \end{aligned}$$

Elementary row operations

define the **augmented matrix** of the linear equations on page 2-2 as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

the following operations on the row of the augmented matrix:

1. multiply a row through by a nonzero constant
2. interchange two rows
3. add a constant times one row to another

do not alter the solution set and yield a simpler system

these are called **elementary row operations** on a matrix

example:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ -x_1 + x_2 + x_3 &= -1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \overset{\text{augmented matrix}}{\Longrightarrow} \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ -1 & 1 & 1 & -1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add the first row to the second ($R_1 + R_2 \to R_2$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 2 & -1 & -2 & 3 \end{bmatrix}$$

add -2 times the first row to the third ($-2R_1 + R_3 \to R_3$)

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ 4x_2 + 3x_3 &= 1 \\ -7x_2 - 6x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 4 & 3 & 1 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

multiply the second row by $1/4$ ($R_2/4 \rightarrow R_2$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ -7x_2 - 6x_3 & = & -1 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & -7 & -6 & -1 \end{bmatrix}$$

add 7 times the second row to the third ($7R_2 + R_3 \rightarrow R_3$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ -\frac{3}{4}x_3 & = & \frac{3}{4} \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & -3/4 & 3/4 \end{bmatrix}$$

multiply the third row by $-4/3$ ($-4R_3/3 \rightarrow R_3$)

$$\begin{array}{rcl} x_1 + 3x_2 + 2x_3 & = & 2 \\ x_2 + \frac{3}{4}x_3 & = & \frac{1}{4} \\ x_3 & = & -1 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 3/4 & 1/4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add $-3/4$ times the third row to the second $(R_2 - (3/4)R_3 \to R_2)$

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 2 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add -3 times the second row to the first $(R_1 - 3R_2 \to R_1)$

$$\begin{aligned} x_1 + 2x_3 &= -1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

add -2 times the third row to the first $(R_1 - 2R_2 \to R_1)$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Gaussian Elimination

- a systematic procedure for solving systems of linear equations
- based on performing row operations of the augmented matrix

Definition: a matrix is in **row echelon form** if

1. a row does not consist entirely of zeros, then the first nonzero number in the row is a 1 (called a leading 1)
2. all nonzero rows are above any rows of all zeros
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row

examples:

$$\begin{bmatrix} 1 & 4 & -3 & 5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition: a matrix is in **reduced row echelon form** if

- it is in a row echelon form and
- every leading 1 is the only nonzero entry in its colum

examples:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Facts about echelon forms

1. every matrix has a *unique* reduced row echelon form
2. row echelon forms are not unique
3. all row echelon forms of a matrix have the same number of zero rows
4. the leading 1's always occur in the same positions in the row echelon forms of a matrix A

   those positions are callled the **pivot** positions of A

   a column that contains a pivot position is called a **pivot column** of A.

Inspecting a solution

- simplify the augmented matrix to the *reduced echelon form*
- read the solution from the reduced echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies 0 \cdot x_3 = 1 \quad \text{(no solution)}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix} \implies x_1 = -2, \ \ x_2 = -1, \ \ x_3 = 5 \quad \text{(unique solution)}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies x_1 = 2, \ x_2 = 1 \quad \text{(unique solution)}$$

another example

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies \begin{array}{rcr} x_1 + 3x_2 & = & -2 \\ x_2 - x_3 & = & 1 \end{array}$$

Definition:

- the corresponding variables to the leading 1's are called **leading variables**
- the remaining variables are called **free variables**

here x_1, x_2 are leading variables and x_3 is a free variable

let $x_3 = t$ and we obtain

$$x_1 = -3t - 2, \quad x_2 = t + 1, \quad x_3 = t$$

(many solutions)

$$\begin{bmatrix} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \implies x_1 - 5x_2 + x_3 = 4$$

x_1 is the leading variable, x_2 and x_3 are free variables

let $x_2 = s$ and $x_3 = t$ we obtain

$$\begin{aligned} x_1 &= 5s - t + 4 \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

(many solutions)

by assigning values to s and t, a set of parametric equations:

$$\begin{aligned} x_1 &= 5s - t + 4 \\ x_2 &= s \\ x_3 &= t \end{aligned}$$

is called a **general solution** of the system

Gaussian-Jordan elimination

- simplify an augmented matrix to the reduced row echelon form
- inspect the solution from the reduced row echelon form
- the algorithm consists of two parts:
  - **forward phase:** zeros are introduced below the leading 1's
  - **backward phase:** zeros are introduced above the leading 1's

example:

$$\begin{array}{rcl} x_1 + x_2 + 2x_3 & = & 8 \\ -x_1 - 2x_2 + 3x_3 & = & 1 \\ 3x_1 - 7x_2 + 4x_3 & = & 10 \end{array} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

use row operations

$$\begin{array}{ccc} R_1 + R_2 \to R_2 & -3R_1 + R_3 \to R_3 & (-1) \cdot R_2 \to R_2 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 3 & -7 & 4 & 10 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \end{array}$$

$$\begin{array}{cc} 10R_2 + R_3 \to R_3 & R_3/(-52) \to R_3 \\ \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ & \text{(a row echelon form)} \end{array}$$

we have added zero below the leading 1's (forward phase) continue performing row operations

$$
\begin{array}{ccc}
5R_3 + R_2 \to R_2 & -R_2 + R_1 \to R_1 & -2R_3 + R_1 \to R_1 \\
\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} &
\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
\end{array}
$$

(reduced echelon form)

we have added zero above the leading 1's (backward phase)

the system has a unique solution

$$x_1 = 3, \quad x_2 = 1, \quad x_3 = 2$$

Homogeneous linear systems

Definition:

a system of linear equations is said to be **homogeneous** if b_j's are all zero

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

- $x_1 = x_2 = \cdots = x_n = 0$ is the **trivial** solution
- if $(x_1, x_2, \ldots, x_n)$ is a solution, so is $(\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ for any $\alpha \in \mathbf{R}$
- hence, if a solution exists, then the system has infinitely many solutions (by varying α)

more properties

- the last column of the augmented matrix is entirely zero
- the zero columns do not alter under any row operations, so the linear systems corresponding to the reduced echelon form is homogeneous
- if the reduced row echelon form has r *nonzero* rows, then the system has $n - r$ free variables
- a homogeneous linear system with more unknowns than equations has infinitely many solutions

example

$$
\begin{aligned}
x_1 - x_2 + 2x_3 - x_4 &= 0 \\
2x_1 + x_2 - 2x_3 - 2x_4 &= 0 \\
-x_1 + 2x_2 - 4x_3 + x_4 &= 0 \\
3x_1 - 3x_4 &= 0
\end{aligned}
\quad \Longrightarrow \quad
\begin{bmatrix}
1 & -1 & 2 & -1 & 0 \\
2 & 1 & -2 & -2 & 0 \\
-1 & 2 & -4 & 1 & 0 \\
3 & 0 & 0 & -3 & 0
\end{bmatrix}
$$

the reduced echelon form is

$$
\begin{bmatrix}
1 & 0 & 0 & -1 & 0 \\
0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad \Longrightarrow \quad
\begin{aligned}
x_1 - x_4 &= 0 \\
x_2 - 2x_3 &= 0
\end{aligned}
$$

define $x_3 = s, x_4 = t$, the parametric equation is

$$x_1 = t, \quad x_2 = 2s, \quad x_3 = s, \quad x_4 = t$$

there are two nonzero rows, so we have two $(n - 2 = 2)$ free variables

MATLAB commands

`rref(A)` produces the reduced row echelon form of a matrix A

```
>> A = [-1 2 4 1;0 1 2 1;2 3 6 5]
A =
    -1     2     4     1
     0     1     2     1
     2     3     6     5

>> rref(A)

ans =

     1     0     0     1
     0     1     2     1
     0     0     0     0
```

References

Chapter 1 in

H. Anton, *Elementary Linear Algebra*, 10th edition, Wiley, 2010