11. Integrals

- derivatives of functions
- definite integrals
- contour integrals
- Cauchy-Goursat theorem
- Cauchy integral formula

Derivatives of functions

consider derivatives of complex-valued functions w of a *real* variable t

$$w(t) = u(t) + jv(t)$$

where u and v are **real-valued** functions of t

the **derivative** $w'(t)$ or $\frac{d}{dt}w(t)$ is defined as

$$w'(t) = u'(t) + jv'(t)$$

Properties ✎ many rules are carried over to complex-valued functions

- $[cw(t)]' = cw'(t)$
- $[w(t) + s(t)]' = w'(t) + s'(t)$
- $[w(t)s(t)]' = w'(t)s(t) + w(t)s'(t)$

mean-value theorem: no longer applies for complex-valued functions

suppose $w(t)$ is continuous on $[a, b]$ and $w'(t)$ exists

it is *not necessarily true* that there is a number $c \in [a, b]$ such that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for example, $w(t) = e^{jt}$ on the interval $[0, 2\pi]$ and we have $w(2\pi) - w(0) = 0$

however, $|w'(t)| = |je^{jt}| = 1$, which is never zero

Definite integrals

the **definite integral** of a complex-valued function

$$w(t) = u(t) + jv(t)$$

over an interval $a \leq t \leq b$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + j\int_a^b v(t)dt$$

provided that each integral exists (ensured if u and v are piecewise continuous)

Properties ✎

- $\int_a^b [cw(t) + s(t)]dt = c\int_a^b w(t)dt + \int_a^b s(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

Fundamental Theorem of Calculus: still holds for complex-valued functions

suppose

$$W(t) = U(t) + jV(t) \quad \text{and} \quad w(t) = u(t) + jv(t)$$

are *continuous* on $[a, b]$

if $W'(t) = w(t)$ when $a \leq t \leq b$ then $U'(t) = u(t)$ and $V'(t) = v(t)$

then the integral becomes

$$\int_a^b w(t)dt = U(t)\Big|_a^b + j\, V(t)\Big|_a^b = [U(b) + jV(b)] - [U(a) + jV(a)]$$

therefore, we obtain

$$\int_a^b w(t)dt = W(b) - W(a)$$

example: compute $\int_0^{\pi/6} e^{j2t} dt$

since

$$\frac{d}{dt}\left(\frac{e^{j2t}}{j2}\right) = e^{j2t}$$

the integral is given by

$$\begin{aligned}\int_0^{\pi/6} e^{j2t} dt &= \frac{1}{j2} e^{j2t}\Big|_0^{\pi/6} \\ &= \frac{1}{j2}[e^{j\pi/3} - e^{j0}] \\ &= \frac{\sqrt{3}}{4} + \frac{j}{4}\end{aligned}$$

mean-value theorem for integration: not hold for complex-valued $w(t)$

it is *not necessarily true* that there exists $c \in [a, b]$ such that

$$\int_a^b w(t)dt = w(c)(b - a)$$

for example, $w(t) = e^{jt}$ for $0 \leq t \leq 2\pi$ (same example as on page 11-3)

it is easy to see that

$$\int_a^b w(t)dt = \int_0^{2\pi} e^{jt}dt = \left.\frac{e^{jt}}{j}\right|_0^{2\pi} = 0$$

but there is no $c \in [0, 2\pi]$ such that $w(c) = 0$

Contour integral

integrals of complex-valued functions defined on **curves** in the complex plane

- arcs
- contours
- contour integrals

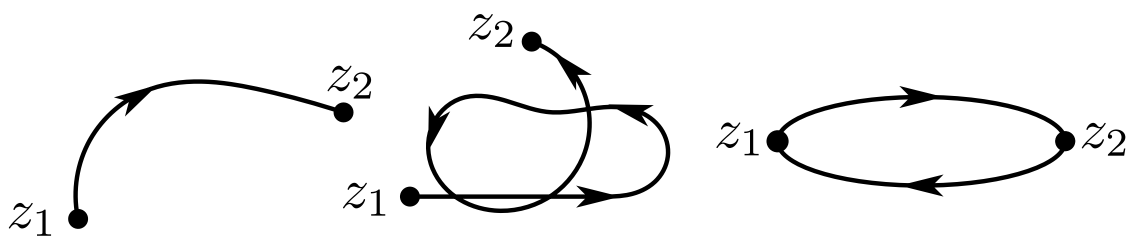
Arcs

a set of points $z = (x, y)$ in the complex plane is said to be an **arc** or a **path** if

$$x = x(t), \quad y = y(t), \text{ or } z(t) = x(t) + jy(t), \qquad a \leq t \leq b$$

where $x(t)$ and $y(t)$ are *continuous* functions of real parameter t

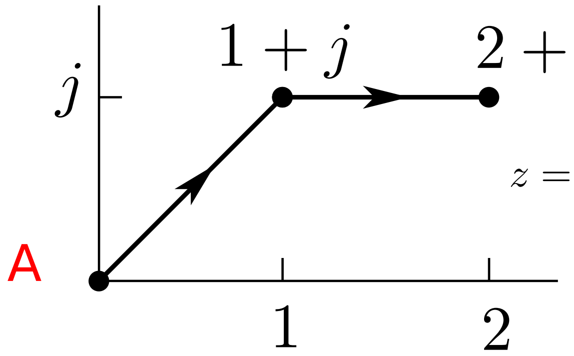
- the arc is **simple** or is called a **Jordan arc** if it does not cross itself, $e.g.$, $z(t) \neq z(s)$ when $t \neq s$
- the arc is **closed** if it starts and ends at the same point, $e.g.$, $z(b) = z(a)$
- a **simple closed path (or curve)** is a *closed* path such that $z(t) \neq z(s)$ for $a \leq s < t < b$



simple arc arc (not simple) simple closed curve

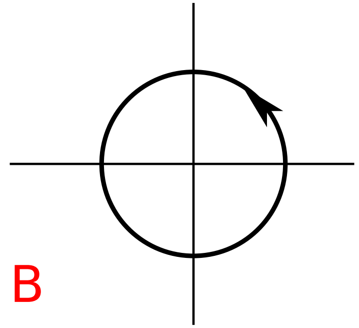
examples:

polygonal line



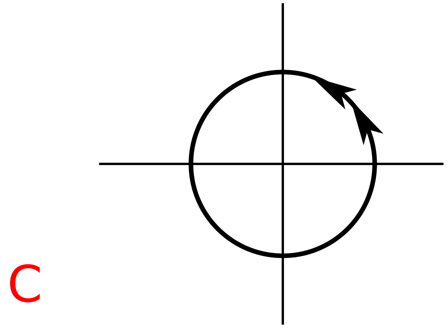
$$z = \begin{cases} x + jx, & 0 \leq x \leq 1 \\ x + j, & 1 \leq x \leq 2 \end{cases}$$

counterclockwise



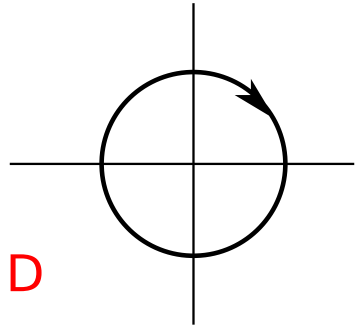
$$z = e^{j\theta}$$
$$0 \leq \theta \leq 2\pi$$

traverse twice



$$z = e^{j2\theta}$$
$$0 \leq \theta \leq 2\pi$$

clockwise



$$z = e^{-j\theta}$$
$$0 \leq \theta \leq 2\pi$$

the arcs B, C and D have the same set of points, but they are *not* the same arc

remark: a closed curve is **positive oriented** if it is counterclockwise direction

Contours

an arc is called **differentiable** if the components $x'(t)$ and $y'(t)$ of the derivative

$$z'(t) = x'(t) + jy'(t)$$

of $z(t)$ used to represent the arc, are **continuous** on the interval $[a, b]$

the arc $z = z(t)$ for $a \leq t \leq b$ is said to be **smooth** if

- $z'(t)$ is continuous on the closed interval $[a, b]$
- $z'(t) \neq 0$ throughout the open interval $a < t < b$

a concatenation of smooth arcs is called a **contour** or **piecewise smooth arc**

Contour integrals

let C be a contour extending from a point a to a point b

an integral defined in terms of the values $f(z)$ along a contour C is denoted by

- $\int_C f(z)dz$ (its value, in general, depends on C)
- $\int_a^b f(z)dz$ (if the integral is *independent* of the choice of C)

if we assume that f is **piecewise continuous** on C then we define

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

as the **line integral** or **contour integral** of f along C in terms of parameter t

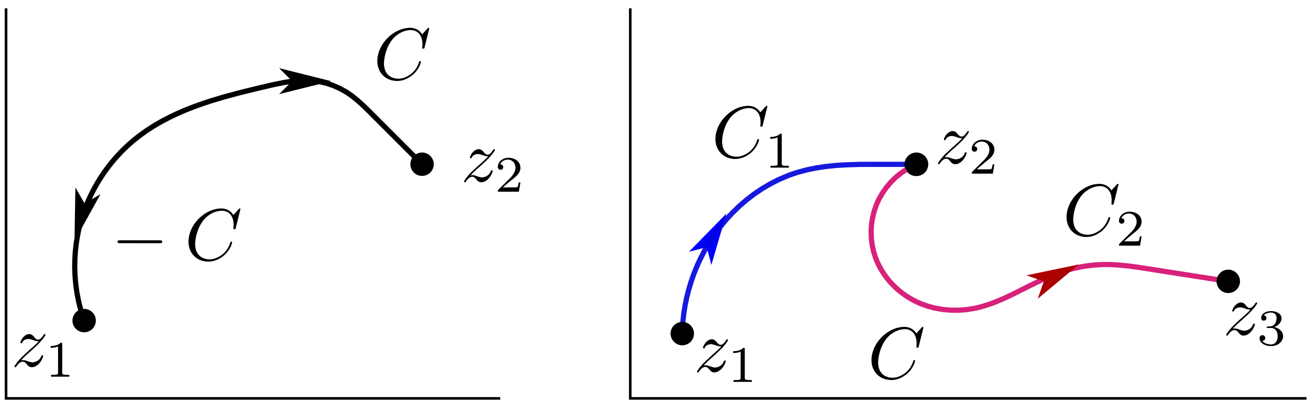
Properties ✎

- $\int_C [z_0 f(z) + g(z)] dz = z_0 \int_C f(z) dz + \int_C g(z) dz, \quad z_0 \in \mathbf{C}$

- $\int_{-C} f(z) dz = -\int_C f(z) dz$

- $\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$

- if C is a simple closed path then we write $\int_C f(z) dz = \oint_C f(z) dz$

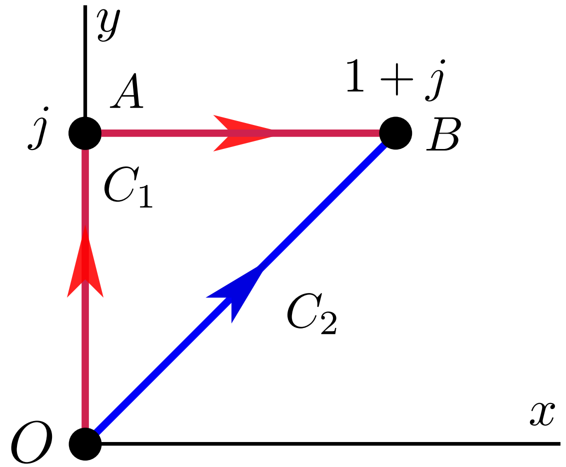


example: $f(z) = y - x - j3x^2 \quad (z = x + jy)$

- $I_1 = \int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$
  - segment OA: $z = 0 + jy, dz = jdy$

    $\int_{OA} f(z)dz = \int_0^1 (y - 0 - j0)jdy = j/2$
  - segment AB: $z = x + j, dz = dx$

    $\int_{AB} f(z)dz = \int_0^1 (1-x-j3x^2)dx = 1/2-j$



- $I_2 = \int_{C_2} f(z)dz$

$$z = x+jx, \quad dz = (1+j)dx, \quad \int_{C_2} f(z)dz = \int_0^1 (x-x-j3x^2)(1+j)dx = 1-j$$

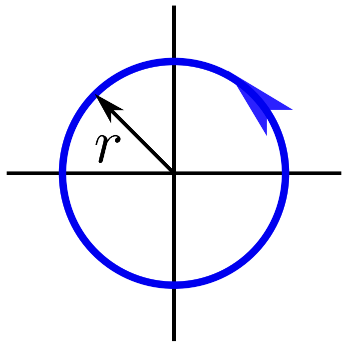
remark: $I_1 = \frac{1-j}{2} \neq I_2$ though C_1 and C_2 start and end at the same points

example: compute $\int_C \bar{z} dz$ on the following contours

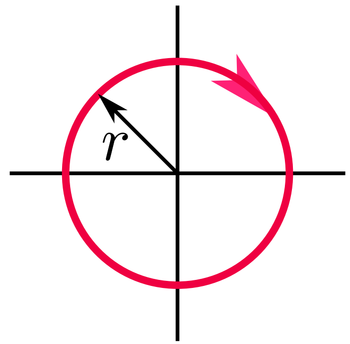
the contour is a circle, so we write z in polar form, and note that r is unchanged

$$z = re^{j\theta}, \quad dz = jre^{j\theta} d\theta, \quad \theta_1 \leq \theta \leq \theta_2$$

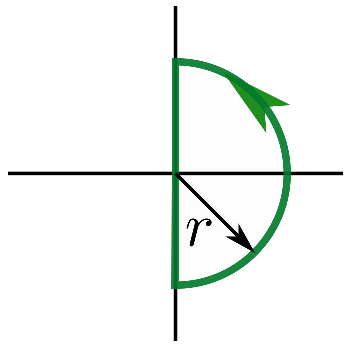
$$I = \int_{\theta_1}^{\theta_2} \overline{re^{j\theta}} \cdot jre^{j\theta} d\theta = jr^2 \int_{\theta_1}^{\theta_2} 1 \; d\theta$$



$$I = jr^2 \int_0^{2\pi} 1 \; d\theta$$
$$= j2\pi r^2$$

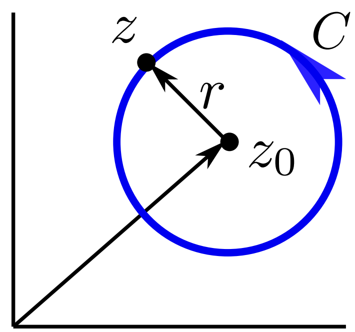


$$I = -jr^2 \int_0^{2\pi} 1 \; d\theta$$
$$= -j2\pi r^2$$



$$I = jr^2 \int_{-\pi/2}^{\pi/2} 1 \; d\theta$$
$$= j\pi r^2$$

example: let C be a circle of radius r, centered at z_0



show that $$\int_C (z - z_0)^m dz = \begin{cases} 0, & m \neq -1 \\ j2\pi, & m = -1 \end{cases}$$

we parametrize the circle by writing

$$z = z_0 + re^{j\theta}, \quad 0 \leq \theta \leq 2\pi, \quad \text{so } dz = jre^{j\theta} d\theta$$

the integral becomes

$$I = \int_C r^m e^{jm\theta} \cdot jre^{j\theta} dz = jr^{m+1} \int_0^{2\pi} e^{j(m+1)\theta} d\theta$$

if $m = -1$, $I = j \int_0^{2\pi} d\theta = j2\pi$; otherwise, for $m \neq -1$, we have

$$I = jr^{m+1} \int_0^{2\pi} \{\cos[(m+1)\theta] + j\sin[(m+1)\theta]\} \; d\theta = 0$$

Independence of path

under which condition does a contour integral only depend on the endpoints ?

assumptions:

- let D be a domain and $f : D \to \mathbf{C}$ be a continuous function
- let C be *any contour* in D that starts from z_1 to z_2

we say f has an **antiderivative** in D if there exists $F : D \to \mathbf{C}$ such that

$$F'(z) = \frac{dF(z)}{dz} = f(z)$$

Theorem: if f has an antiderivative F on D, the contour integral is given by

$$\int_C f(z)dz = F(z_2) - F(z_1)$$

example: $f(z)$ is the principal branch

$$z^j = e^{j\,\mathrm{Log}\,z} \quad (|z| > 0, -\pi < \mathrm{Arg}\,z < \pi)$$

of this power function, compute the integral

$$\int_{-1}^{1} z^j dz$$

by two methods:

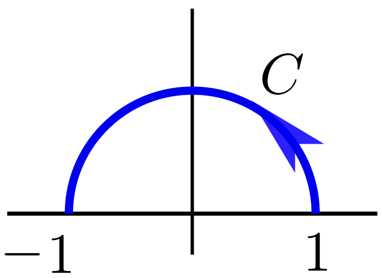
- using a parametrized curve C which is the semicircle $z = e^{j\theta}$, $(0 \leq \theta \leq \pi)$
- using an antiderivative of f of the branch

$$z^j = e^{j\log z} \quad (|z| > 0, -\pi/2 < \arg z < 3\pi/2)$$

parametrized curve: $z = e^{j\theta}$ and $dz = je^{j\theta}d\theta$

$$z^j = e^{j \log z} = e^{j(\mathrm{Log}\, 1 + j \arg z)} = e^{j \cdot j\theta} = e^{-\theta}, \quad (0 < \theta < \pi)$$

the integral becomes



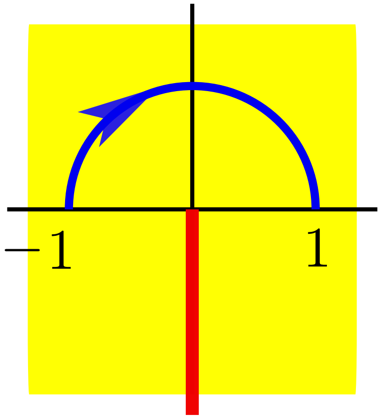
$$\begin{aligned} \int_C z^j dz &= \int_0^\pi je^{(j-1)\theta} d\theta \\ &= \frac{j}{j-1}\, e^{(j-1)\theta}\Big|_0^\pi \\ &= \frac{j}{j-1}(e^{(j-1)\pi} - 1) = \frac{-j}{j-1}(e^{-\pi} + 1) \\ &= -\frac{(1-j)(e^{-\pi}+1)}{2} \end{aligned}$$

hence, $\int_{-1}^{1} z^j dz = \int_{-C} z^j dz = \dfrac{(1-j)(e^{-\pi 1}+1)}{2}$

antiderivative of z^j is $z^{j+1}/(j+1)$ on the branch

$$z^j = e^{j\log z} \quad (|z| > 0, -\pi/2 < \arg z < 3\pi/2)$$

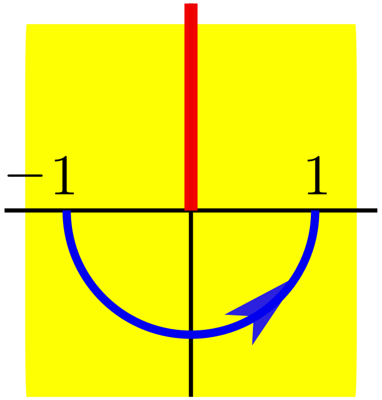
(we cannot use the principal branch because it is not defined at $z = -1$)



$$\begin{aligned}\int_{-1}^{1} z^j dz &= \left[\frac{z^{j+1}}{j+1}\right]_{-1}^{1} = \frac{1}{j+1}\left[1^{j+1} - (-1)^{j+1}\right] \\ &= \frac{1}{j+1}\left[e^{(j+1)\log 1} - e^{(j+1)\log(-1)}\right] \\ &= \frac{1}{j+1}\left[e^{(j+1)(\operatorname{Log} 1 + j0)} - e^{(j+1)(\operatorname{Log} 1 + j\pi)}\right] \\ &= \frac{1}{j+1}\left[1 - e^{j\pi - \pi}\right] = \frac{(1-j)(e^{-\pi}+1)}{2}\end{aligned}$$

the integral computed by the two methods are equal if we use an antiderivative of z^j on a different branch

$$z^j = e^{j \log z} \quad (|z| > 0, \pi/2 < \arg z < 5\pi/2)$$



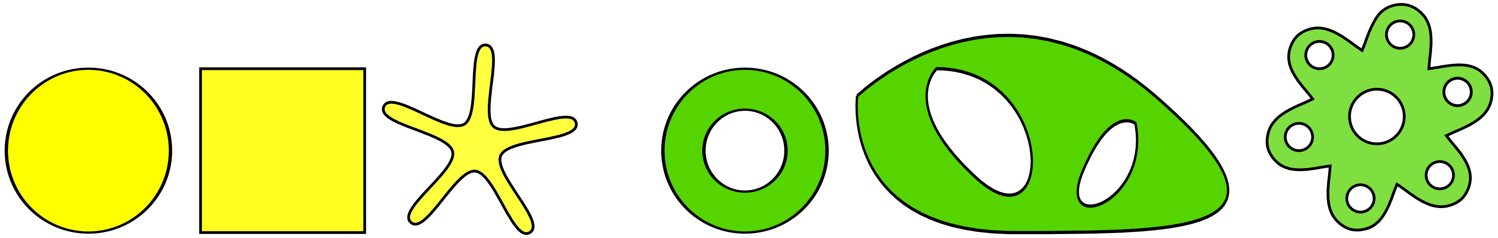
$$\begin{aligned}
\int_{-1}^{1} z^j dz &= \frac{1}{j+1}\left[e^{(j+1)\log 1} - e^{(j+1)\log(-1)}\right] \\
&= \frac{1}{j+1}\left[e^{(j+1)(\mathrm{Log}\, 1 + j2\pi)} - e^{(j+1)(\mathrm{Log}\, 1 + j\pi)}\right] \\
&= \frac{1}{j+1}\left[e^{-2\pi + j2\pi} - e^{-\pi + j\pi}\right] \\
&= \frac{1}{j+1}\left[e^{-2\pi} + e^{-\pi}\right] \\
&= \frac{(1-j)e^{-\pi}(e^{-\pi}+1)}{2}
\end{aligned}$$

the integral is different now as the function value of the integrand has changed

Simply and Multiply connected domains

a **simply connected** domain D is a domain such that every simple closed contour within it encloses only points of D

intuition: a domain is simply connected if it has no **holes**



simply connected   multiply connected

a domain that is not simply connected is called **multiply connected**

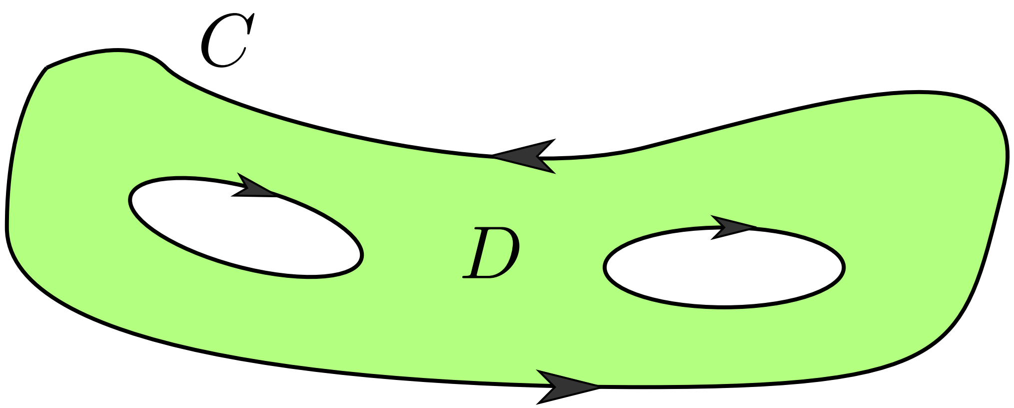
Green's Theorem

let D be a bounded domain whose boundary C is sectionally smooth

let $P(x,y)$ and $Q(x,y)$ be *continuously differentiable* on $D \cup C$, then

$$\int_C P dx + \int_C Q dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where C is in the positive direction w.r.t. the interior of D



D can be simply or multiply connected

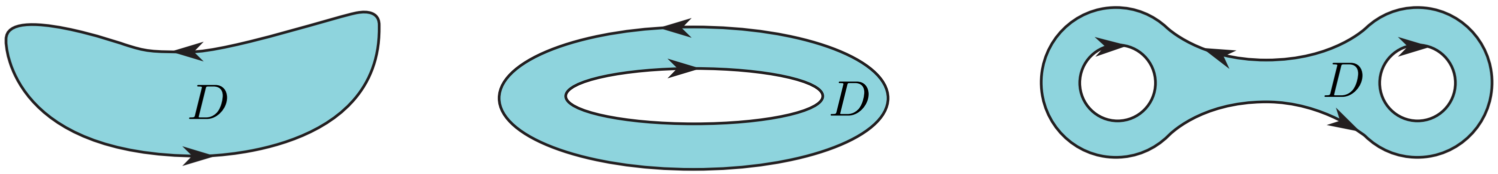
this result will be used to prove the Cauchy's theorem

Cauchy's theorem

let D be a bounded domain whose boundary C is sectionally smooth

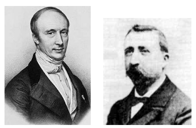
Theorem: if $f(z)$ is analytic and $f'(z)$ is continuous **in** D **and on** C then

$$\int_C f(z)dz = 0$$



Goursat proved this result w/o the assumption on continuity of f'

the consequence is then known as the **Cauchy-Goursat theorem**



Proof of Cauchy's theorem:

$$f(z) = u(x,y) + jv(x,y), \quad dz = dx + jdy$$
$$f(z)dz = (u + jv)(dx + jdy) = u\ dx - v\ dy + j(v\ dx + u\ dy)$$

if f' is continuous in D, so are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$, then from Green's theorem

$$\int_C f(z)dz = \int\int_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + j\int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy$$

since f is analytic, the Cauchy-Riemann equations suggest that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so we can conclude that

$$\int_C f(z)dz = 0$$

example: for *any* simple closed contour C

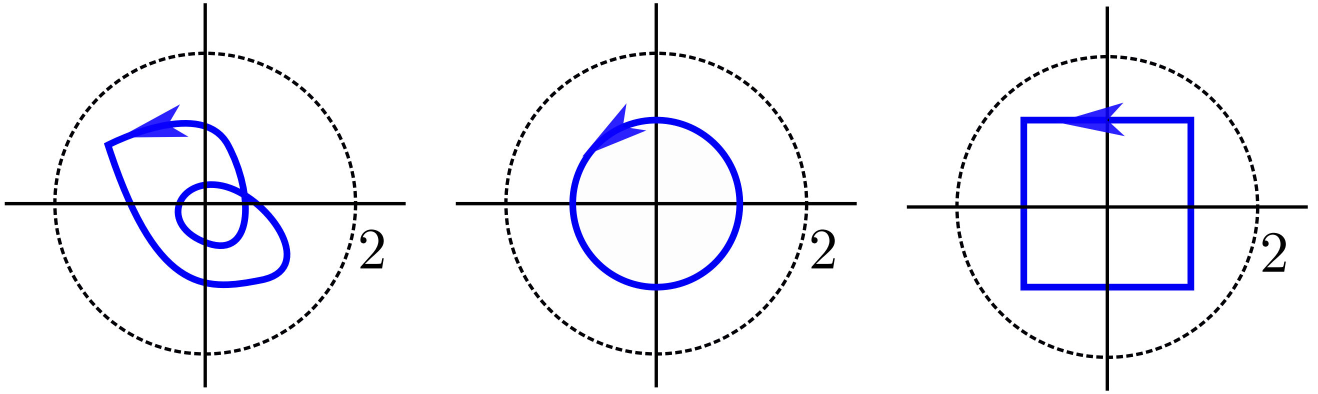
$$\int_C e^{z^2} dz = 0$$

because e^{z^2} is a composite of e^z and z^2, so f is analytic everywhere

example: the integral

$$\int_C \frac{ze^z}{(z^2+4)^2} dz = 0$$

for any closed contour lying in the open disk $|z| < 2$



Extension to multiply connected domains

let D be a *multiply connected domain*

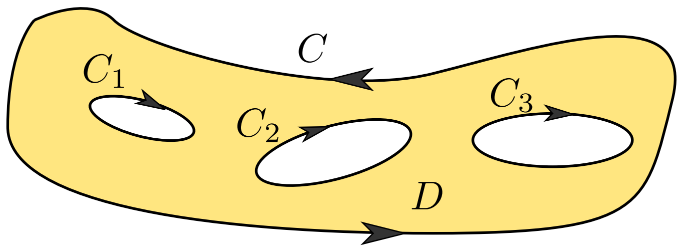
Cauchy-Goursat theorem: suppose that

1. C is a simple closed contour in D, described in **counterclockwise** direction
2. $C_1, \ldots, C_n$ are simple closed contours interior to C, all in **clockwise** direction
3. $C_1, \ldots, C_n$ are **disjoint** and their interiors have no points in common

(then D consists of the points in C and exterior to each C_k)

if f is analytic **on** *all of these contours* and **throughout** D then

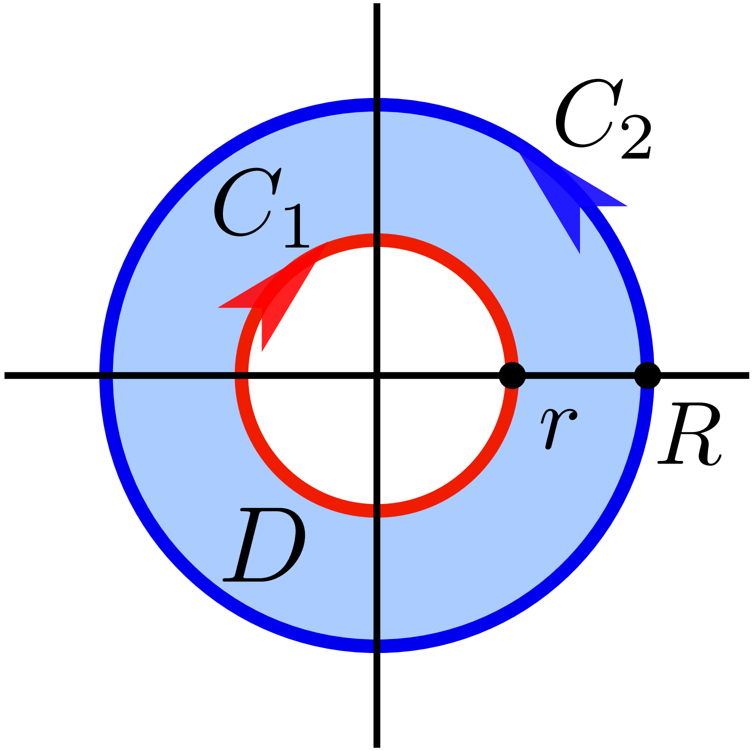
$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$



example: use the result from page 11-16 to compute

$$\int_C \frac{1}{z}\, dz$$

where C is the boundary of the annulus D shown below (where $r, R > 0$)



if $z = r^{j\theta}$ then $z\bar{z} = |z|^2 = r^2$

from p. 11-16, we obtain

$$\int_{C_2} R^2/z dz = j2\pi R^2, \quad \text{or that}$$

$$\int_{C_1} \frac{1}{z} dz = -j2\pi, \quad \int_{C_2} \frac{1}{z} dz = j2\pi$$

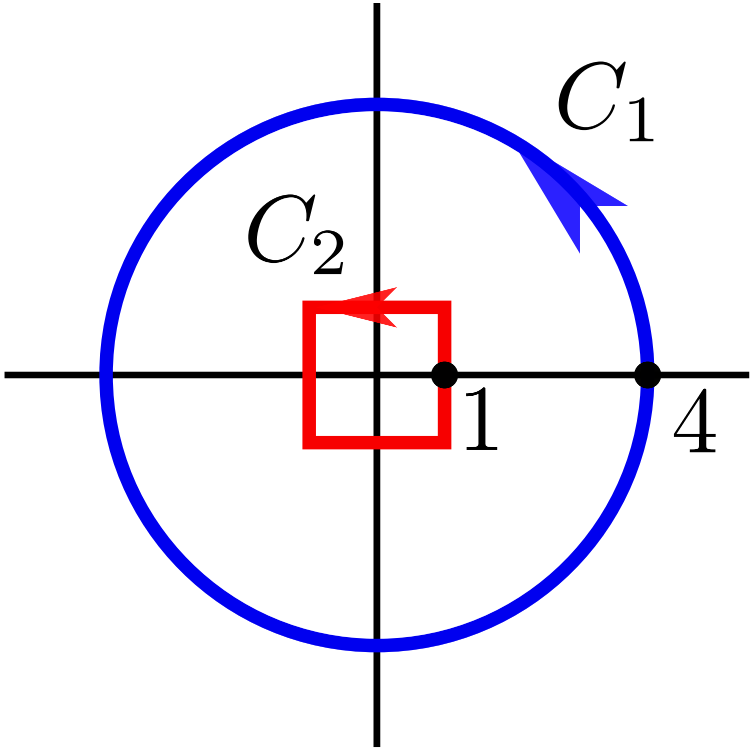
therefore, $\int_C \frac{1}{z}\, dz = \int_{C_1} \frac{1}{z}\, dz + \int_{C_2} \frac{1}{z} dz = 0$

agree with the Cauchy's theorem since f is analytic everywhere in D and on C

example: for each f, use the Cauchy-Goursat theorem on p. 11-27 to show that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

where C_1 is a circle with radius 4 and C_2 is a square shown below



$$f(z) = \frac{1}{3z^2+1}$$

$$f(z) = \frac{z+2}{\sin(z/2)}$$

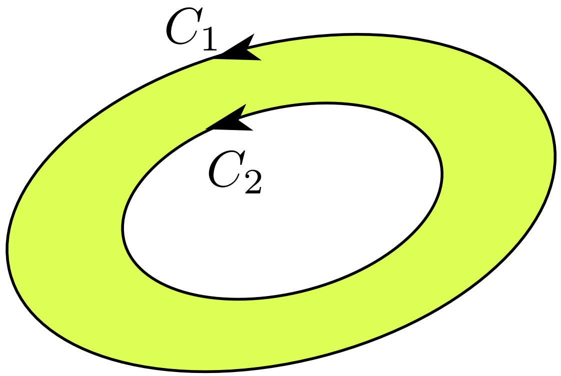
$$f(z) = \frac{z}{1-e^z}$$

where are the singular points of these f?

this result is known as **the principle of deformation path**

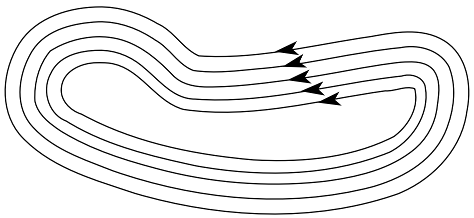
Principle of deformation of paths

let C_1 and C_2 be **positively oriented** simple closed contours where C_2 is interior to C_1



Theorem: if a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$



meaning: integrals of an analytic function does **not depend** on the path if the function is analytic in *between* and *on* the two paths

Cauchy integral formula

let C be a simple closed contour, taken in the positive sense

Theorem: let f be analytic everywhere *inside* and *on* C

if z_0 is any point interior to C then

$$j2\pi f(z_0) = \int_C \frac{f(z)}{(z - z_0)} dz$$

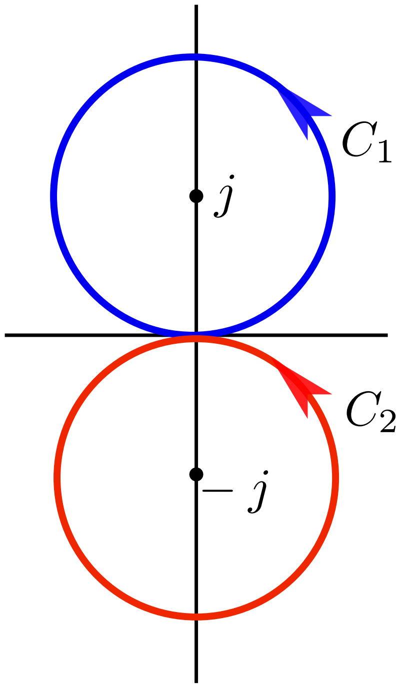
this is known as the **Cauchy integral formula**

meaning: certain integrals along contours can be determined by the values of f

example: compute $\int_C \dfrac{e^z}{z^2+1}\,dz$ on the contours C_1 and C_2

$$\text{write} \quad \int_C \frac{e^z}{z^2+1} dz = \int_C \frac{e^z}{(z+j)(z-j)} dz$$

choose $f(z)$ such that it is analytic everywhere on each contour



- to compute $\int_{C_1} \frac{e^z}{z^2+1} dz$ choose $f(z) = e^z/(z+j)$

$$\int_{C_1} \frac{e^z}{(z+j)(z-j)} dz = j2\pi f(j) = \pi e^j$$

- to compute $\int_{C_2} \frac{e^z}{z^2+1} dz$ choose $f(z) = e^z/(z-j)$

$$\int_{C_1} \frac{e^z}{(z+j)(z-j)} dz = j2\pi f(-j) = -\pi e^{-j}$$

Upper bounds for contour integrals

assumptions:

- C denotes a contour of length L
- f is piecewise continuous on C

✎ **Theorem:** if there exists a constant $M > 0$ such that

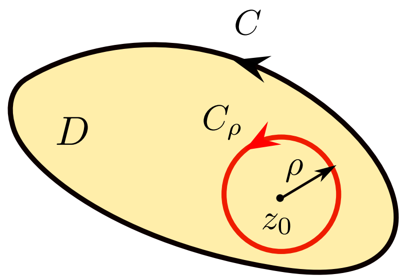
$$|f(z)| \leq M$$

for all z on C at which $f(z)$ is defined, then

$$\left| \int_a^b f(z) dz \right| \leq ML$$

Proof of Cauchy integral formula

create a small circle C_ρ which is interior to C



$f(z)$ is analytic everywhere in D

$\dfrac{f(z)}{z - z_0}$ is analytic in D except at $z = z_0$

from the Cauchy-Goursat theorem,

$$\int_C \frac{f(z)}{z - z_0} dz = \int_{C_\rho} \frac{f(z)}{z - z_0}\, dz$$

which can be expressed as

$$\int_C \frac{f(z)}{z - z_0} dz - f(z_0) \int_{C_\rho} \frac{dz}{z - z_0} = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz$$

we can show that $\int_{C_\rho} \frac{dz}{z - z_0} = j2\pi$ (similar to example on page 11-16)

therefore, we obtain

$$\int_C \frac{f(z)}{z - z_0} dz - j2\pi f(z_0) = \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz$$

and we will show that the RHS must be zero

since f is analytic, it is continuous at z_0, $e.g.$, for each $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

if we pick ρ to be smaller than δ then $|f(z) - f(z_0)|/|z - z_0| < \varepsilon/\rho$

we can show that the integral is bounded by (from page 11-33)

$$\left| \int_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\varepsilon \cdot \text{length of } C_\rho}{\rho} = 2\pi\varepsilon$$

since we can let ε be arbitrarily small, the integral must be equal to zero

Derivatives of analytic functions

let D be a simply connected domain and z_0 be any interior point of D

Theorem: if f is analytic in D then the derivative of $f(z_0)$ of all order exist and are analytic in D

moreover, the derivatives of f at z are given by

$$\frac{j2\pi f^{(n)}(z_0)}{n!} = \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz \qquad (n = 1, 2, \ldots)$$

where C is a closed contour lying on D and z_0 is inside C

example: compute $\int_C \frac{e^{2z}}{z^4} dz$ where C is the positively oriented unit circle

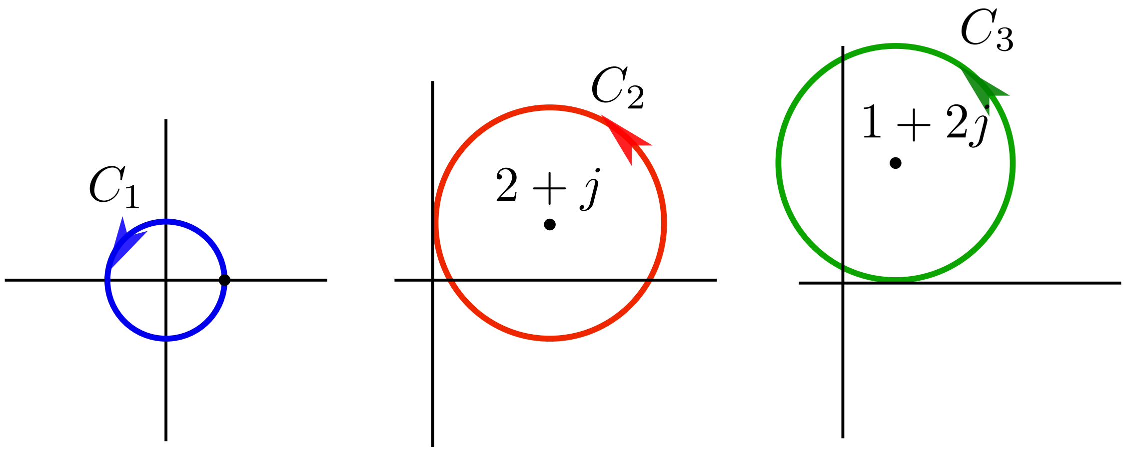
$$\int_C \frac{e^{2z}}{z^4}\,dz \triangleq \int_C \frac{f(z)}{(z-0)^{3+1}} dz = \frac{j2\pi f^{(3)}(0)}{3!} = \frac{j8\pi}{3}$$

(where $f(z) = e^{2z}$)

example: compute $\int_C f(z)dz$ where $f(z) = \dfrac{(z+1)}{(z^3 - 2z^2)}$

C are circles given by $|z| = 1, \qquad |z - 2 - j| = 2, \qquad |z - 1 - j2| = 2$

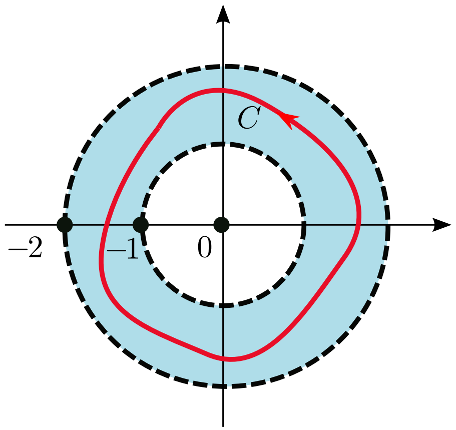
(all are in counterclockwise direction)



$$f_1(z) = \frac{z+1}{z-2}, \qquad f_2(z) = \frac{z+1}{z^2}, \qquad f_3(z) = \frac{z+1}{z^2(z-2)} = f(z)$$

$$\int_{C_1} f(z)\,dz = j2\pi f_1'(0), \; \int_{C_2} f(z)\,dz = j2\pi f_2(2), \; \int_{C_3} f(z)\,dz = 0$$

example: let C be a simple closed contour lying in the annulus $1 < |z| < 2$



compute $$\int_C \frac{3z^3 + 2z^2 - 8z - 4}{z^2(z^2 + 3z + 2)} dz$$

f is not analytic at $0, -1, -2$, so the Cauchy formula cannot be readily applied

we can compute the partial fraction of f and the integral becomes

$$\int_C f(z)dz = -\int_C \frac{1}{z}dz - \int_C \frac{1}{z^2}dz + \int_C \frac{3}{z+1}dz + \int_C \frac{1}{z+2}dz$$

applying the Cauchy integral formula to each term gives

$$\int_C f(z)dz = j2\pi(-1) + j2\pi(0) + j2\pi(3) + 0 = j4\pi$$

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