9. Analytic Functions

- functions of complex variables
- mappings
- limits, continuity, and derivatives
- Cauchy-Riemann equations
- analytic functions

Functions of complex variables

a function f defined on a set S is a rule that assigns a complex number w to each $z \in S$

- S is called the **domain** of definition of f
- w is called the **value** of f at z, denoted by $w = f(z)$
- the domain of f is the set of z such that $f(z)$ is well-defined
- if the value of f is always real, then f is called a **real-valued** function

example: $f(z) = 1/|z|$

- let $z = x + jy$ then $f(z) = 1/(x^2 + y^2)$
- f is a real-valued function
- the domain of f is $\mathbf{C} \backslash \{0\}$

suppose $w = u + jv$ is the value of a function f at $z = x + jy$, so that

$$u + jv = f(x + jy)$$

then we can express f in terms of a pair of real-valued functions of x, y

$$f(z) = u(x, y) + jv(x, y)$$

example: $f(z) = 1/(z^2 + 1)$

- the domain of f is $\mathbf{C} \backslash \{\pm j\}$
- for $z = x + jy$, we can write $f(z) = u(x, y) + jv(x, y)$ by

$$f(x + jy) = \frac{1}{x^2 - y^2 + 1 + j2xy} = \frac{x^2 - y^2 + 1 - j2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2}$$

$$u(x, y) = \frac{x^2 - y^2 + 1}{(x^2 - y^2 + 1)^2 + 4x^2y^2)}, \quad v(x, y) = -\frac{2xy}{(x^2 - y^2 + 1)^2 + 4x^2y^2)}$$

if the polar coordinate r and θ is used, then we can express f as

$$f(re^{j\theta}) = u(r,\theta) + jv(r,\theta)$$

example: $f(z) = z + 1/z$, $z \neq 0$

$$\begin{aligned} f(re^{j\theta}) &= re^{j\theta} + (1/r)e^{-j\theta} \\ &= (r + 1/r)\cos\theta + j(r - 1/r)\sin\theta \end{aligned}$$

Mappings

consider $w = f(z)$ as a *mapping* or a *transformation*

example:

- translation each point z by 1

$$w = f(z) = z + 1 = (x + 1) + jy$$

- rotate each point z by 90°

$$w = f(z) = iz = re^{j(\theta + \pi/2)}$$

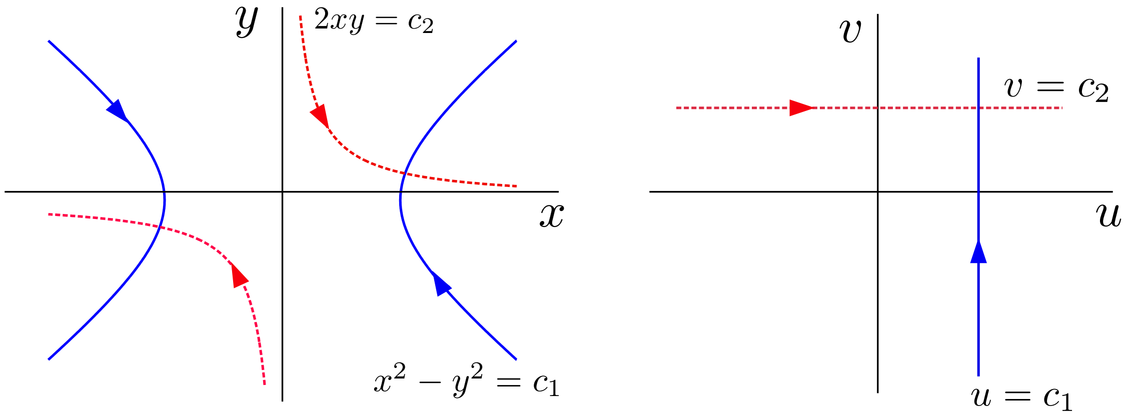
- reflect each point z in the real axis

$$w = f(z) = \bar{z} = x - jy$$

it is useful to sketch images under a given mapping

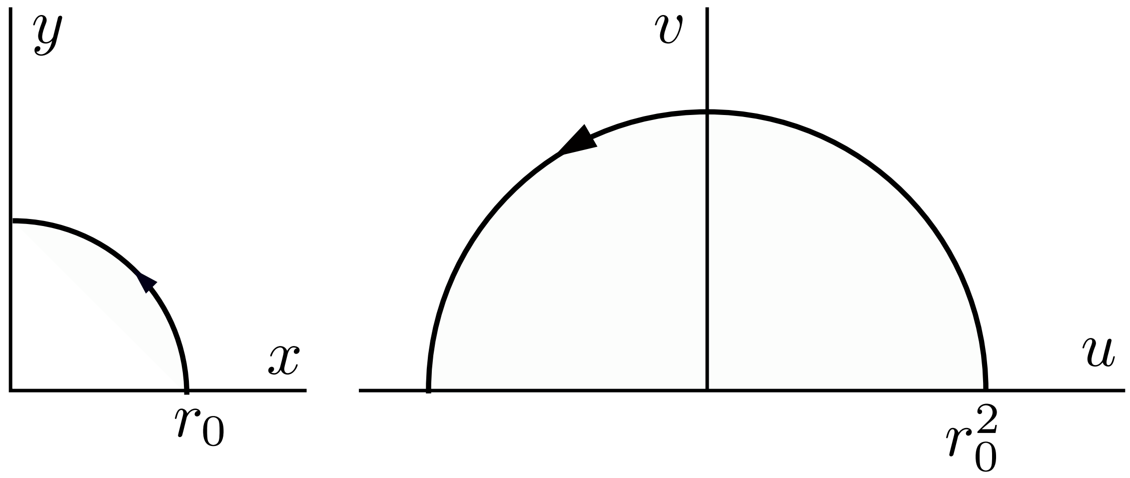
example 1: given $w = z^2$, sketch the image of the mapping on the xy plane

$$w = u(x,y) + jv(x,y), \quad \text{where} \quad u = x^2 - y^2, \quad v = 2xy$$



- for $c_1 > 0$, $x^2 - y^2 = c_1$ is mapped onto the line $u = c_1$
- if $u = c_1$ then $v = \pm 2y\sqrt{y^2 + c_1}$, where $-\infty < y < \infty$
- for $c_2 > 0$, $2xy = c_2$ is mapped into the line $v = c_2$
- if $v = c_2$ then $u = c_2^2/4y^2 - y^2$ where $-\infty < y < 0$, or
- if $v = c_2$ then $u = x^2 - c_2^2/4x^2$, $0 < x < \infty$

example 2: sketch the mapping $w = z^2$ in the polar coordinate



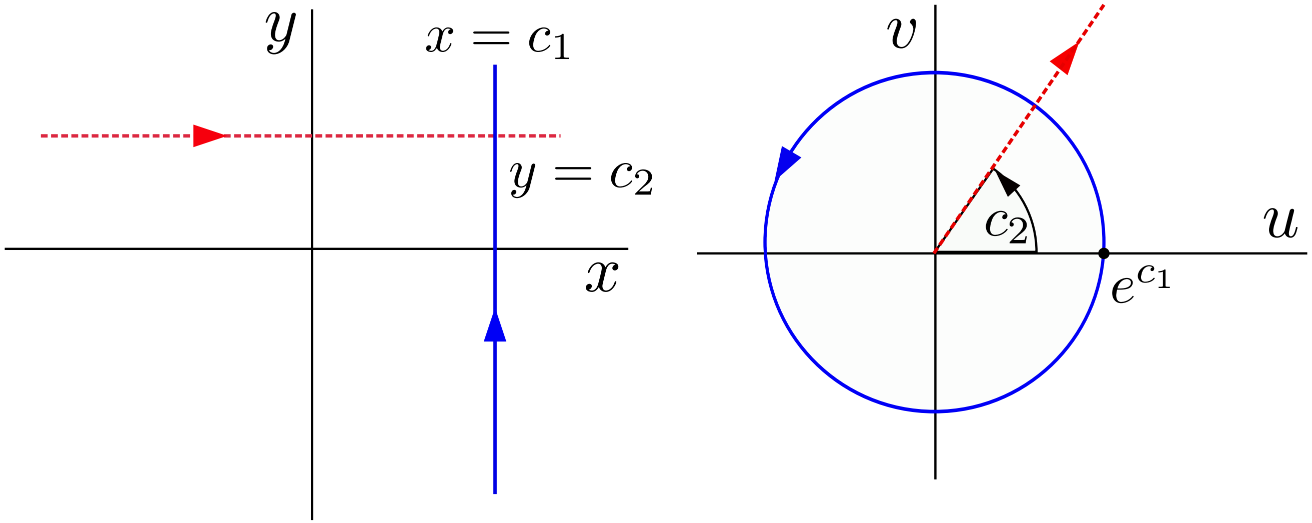
the mapping $w = r^2 e^{j2\phi} = \rho e^{j\theta}$ where

$$\rho = r^2, \quad \theta = 2\phi$$

- the image is found by squaring the modulus and doubling the value θ
- we map the first quadrant onto the upper half plane $\rho \geq 0, 0 \leq \theta \leq \pi$
- we map the upper half plane onto the entire w plane

mappings by the exponential function: $w = e^z$

$$w = e^{x+jy} = \rho e^{j\phi}, \quad \text{where} \quad \rho = e^x, \phi = y$$



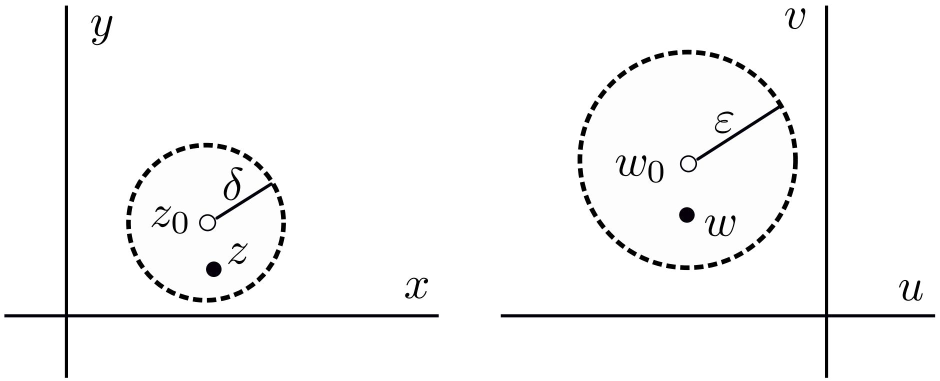
- a vertical line $x = c_1$ is mapped into the circle of radius c_1
- a horizontal line $y = c_2$ is mapped into the ray $\phi = c_2$

Limits

limit of $f(z)$ as z approaches z_0 is a number w_0, $i.e.$,

$$\lim_{z \to z_0} f(z) = w_0$$

meaning: $w = f(z)$ can be made arbitrarily close to w_0 if z is close enough to z_0



Definition: if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

then $w_0 = \lim_{z \to z_0} f(z)$

example: let $f(z) = 2j\bar{z}$, show that $\lim_{z\to 1} f(z) = 2j$

we must show that for *any* $\varepsilon > 0$, we can *always* find $\delta > 0$ such that

$$|z - 1| < \delta \quad \Longrightarrow \quad |2j\bar{z} - 2j| < \varepsilon$$

if we express $|2j\bar{z} - 2j|$ in terms of $|z - 1|$ by

$$|2j\bar{z} - 2j| = 2|\bar{z} - 1| = 2|z - 1|$$

hence if $\delta = \varepsilon/2$ then

$$|f(z) - 2j| = 2|z - 1| < 2\delta < \varepsilon$$

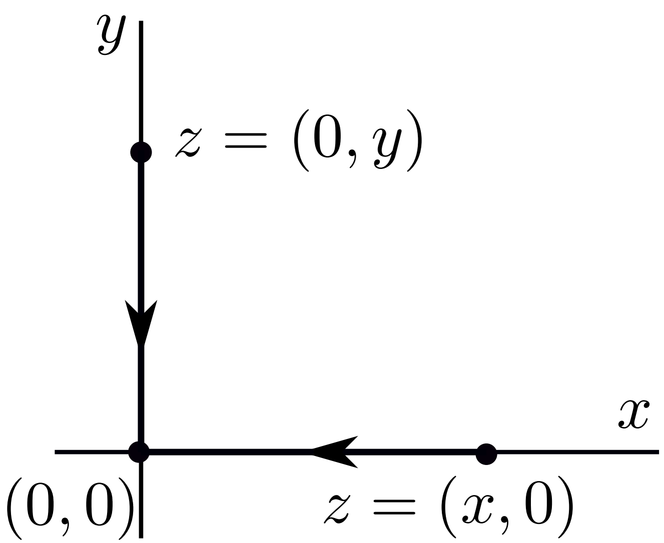
$f(z)$ can be made arbitrarily close to $2j$ by making z close to 1 enough

how close ? determined by δ and ε

Remarks:

- when a limit of $f(z)$ exists at z_0, it is **unique**
- if the limit exists, $z \to z_0$ means z approaches z_0 in any *arbitrary* direction

example: let $f(z) = z/\bar{z}$



- if $z = x$ then $f(z) = \frac{x+j0}{x-j0} = 1$

  as $z \to 0$, $f(z) \to 1$ along the real axis
- if $z = jy$ then $f(z) = \frac{0+jy}{0-jy} = -1$

  as $z \to 0$, $f(z) \to -1$ along the imaginary axis

since a limit must be unique, we conclude that $\lim_{z \to 0} f(z)$ *does not* exist

Theorems on limits

Theorem ✌ suppose $f(z) = u(x,y) + jv(x,y)$ and

$$z_0 = x_0 + jy_0, \quad w_0 = u_0 + jv_0$$

then $\lim_{z \to z_0} f(z) = w_0$ *if and only if*

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$$

Theorem ✌ suppose $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} g(z) = c_0$ then

- $\lim_{z \to z_0} [f(z) + g(z)] = w_0 + c_0$
- $\lim_{z \to z_0} [f(z)g(z)] = w_0 c_0$
- $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = w_0/c_0$ if $c_0 \neq 0$

Limit of polynomial functions: for $p(z) = a_0 + a_1 z + \cdots + a_n z^n$

$$\lim_{z \to z_0} p(z) = p(z_0)$$

Theorem ✎ suppose $\lim_{z \to z_0} f(z) = w_0$ then

- $\lim_{z \to z_0} f(z) = \infty$ if and only if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$
- $\lim_{z \to \infty} f(z) = w_0$ if and only if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
- $\lim_{z \to \infty} f(z) = \infty$ if and only if $\lim_{z \to 0} \frac{1}{f(1/z)} = 0$

example:

$$\lim_{z \to \infty} \frac{2z + j}{z + 1} = 2 \quad \text{because} \quad \lim_{z \to 0} \frac{(2/z) + j}{(1/z) + j} = \lim_{z \to 0} \frac{2 + jz}{1 + z} = 2$$

Continuity

Definition: f is said to be **continuous** at a point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

provided that both terms must exist

this statement is equivalent to another definition:

$\delta - \varepsilon$ **Definition:** if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

then f is continuous at z_0

example: $f(z) = z/(z^2+1)$

- f is not continuous at $\pm j$ because $f(\pm j)$ do not exist
- f is continuous at 1 because

$$f(1) = 1/2 \quad \text{and} \quad \lim_{z\to 1} \frac{z}{z^2+1} = 1/2$$

example: $f(z) = \begin{cases} \frac{z^2+j3z-2}{z+j}, & z \neq -j \\ 2j, & z = -j \end{cases}$

$$\lim_{z\to -j} f(z) = \lim_{z\to -j} \frac{z^2+j3z-2}{z+j} = \lim_{z\to -j} \frac{(z+j)(z+j2)}{z+j} = \lim_{z\to -j} (z+j2) = j$$

we see that $\lim_{z\to -j} f(z) \neq f(-j) = 2j$

hence, f is not continuous at $z = -j$

Remarks ✎

- f is said to be continuous in a region R if it is continuous at *each point* in R
- if f and g are continuous at a point, then so is $f + g$
- if f and g are continuous at a point, then so is fg
- if f and g are continuous at a point, then so is f/g at any such point if g is not zero there
- if f and g are continuous at a point, then so is $f \circ g$
- $f(z) = u(x, y) + jv(x, y)$ is continuous at $z_0 = (x_0, y_0)$ if and only if

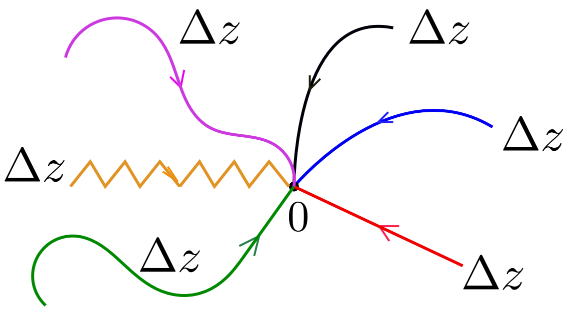
  $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0)

Derivatives

the **complex derivative** of f at z is the limit

$$\frac{df}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

(if the limit exists)



Δz is a complex variable

so the limit must be the same no matter how Δz approaches 0

f is said to be **differentiable** at z when $f'(z)$ exists

example: find the derivative of $f(z) = z^3$

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^3 - z^3}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{3z^2 \Delta z + 3z \Delta z^2 + \Delta z^3}{\Delta z} \\ &= \lim_{\Delta z \to 0} 3z^2 + 3z\Delta z + \Delta z^2 = 3z^2\end{aligned}$$

hence, f is differentiable at any point z and $f'(z) = 3z^2$

example: find the derivative of $f(z) = \bar{z}$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

but $\lim_{z \to 0} z/\bar{z}$ does not exist (page 9-11), so f is not differentiable everywhere

example: $f(z) = |z|^2$ (real-valued function)

$$\begin{aligned}
\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z} = \frac{(z+\Delta z)(\bar{z} + \overline{\Delta z}) - |z|^2}{\Delta z} \\
&= \bar{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \\
&= \begin{cases} \bar{z} + \Delta z + z, & \Delta z = \Delta x + j0 \\ \bar{z} - \Delta z - z, & \Delta z = 0 + j\Delta y \end{cases}
\end{aligned}$$

hence, if $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists then it must be unique, meaning

$$\bar{z} + z = \bar{z} - z \quad \Longrightarrow \quad z = 0$$

therefore f is only differentiable at $z = 0$ and $f'(0) = 0$

note: $f(z) = |z|^2 = u(x,y) + jv(x,y)$ where

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0$$

- f is continuous everywhere because $u(x,y)$ and $v(x,y)$ are continuous
- but f is *not* differentiable everywhere; f' only exists at $z = 0$

hence, for any f we can conclude that

- the continuity of a function *does not* imply the existence of a derivative !
- however, the existence of a derivative *implies* the continuity of f at that point

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \cdot \lim_{z \to z_0} (z - z_0) = f'(z_0) \cdot 0 = 0$$

Theorem ✌ if $f(z)$ is differentiable at z_0 then $f(z)$ is continuous at z_0

Differentiation formulas

basic formulas still hold for complex-valued functions

- $\frac{dc}{dz} = 0$ and $\frac{d}{dz}[cf(z)] = cf'(z)$ where c is a constant
- $\frac{d}{dz}z^n = nz^{n-1}$ if $n \neq 0$ is an integer
- $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
- $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$ (product rule)
- let $h(z) = g(f(z))$ (chain rule)

$$h'(z) = g'(f(z))f'(z)$$

Cauchy-Riemann equations

✌ **Theorem:** suppose that


$$f(z) = u(x, y) + jv(x, y)$$

and $f'(z)$ exists at $z_0 = (x_0, y_0)$ then

- the first-order derivatives of u and v must exist at (x_0, y_0)
- the derivatives must satisfiy the **Cauchy-Riemann equations:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x_0, y_0)$$

and $f'(z_0)$ can be written as

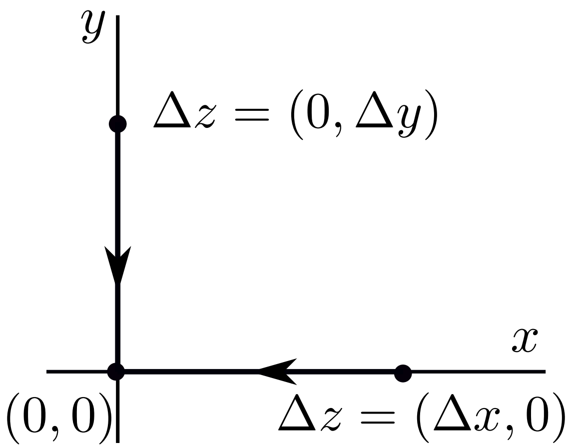
$$f'(z_0) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} \qquad \text{(evaluated at } (x_0, y_0))$$

Proof: we start by writing

$$z = x + jy, \quad \Delta z = \Delta x + j\Delta y$$

and $\Delta w = f(z + \Delta z) - f(z)$ which is

$$\Delta w = u(x + \Delta x, y + \Delta y) - u(x, y) + j[v(x + \Delta x, y + \Delta y) - v(x, y)]$$



- let $\Delta z \to 0$ horizontally $\qquad (\Delta y = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x + \Delta x, y) - u(x, y) + j[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$

- let $\Delta z \to 0$ vertically $\qquad (\Delta x = 0)$

$$\frac{\Delta w}{\Delta z} = \frac{u(x, y + \Delta y) - u(x, y) + j[v(x, y + \Delta y) - v(x, y)]}{j\Delta y}$$

we calculate $f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$ in both directions

- as $\Delta z \to 0$ horizontally

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + j\frac{\partial v}{\partial x}(x, y)$$

- as $\Delta z \to 0$ vertically

$$f'(z) = \frac{\partial v}{\partial y}(x, y) - j\frac{\partial u}{\partial y}(x, y)$$

$f'(z)$ must be valid as $\Delta z \to 0$ in any direction

the proof follows by matching the real/imaginary parts of the two expressions

note: C-R eqs provide **necessary** conditions for the existence of $f'(z)$

example: $f(z) = |z|^2$, we have

$$u(x,y) = x^2 + y^2, \quad v(x,y) = 0$$

if the Cauchy-Riemann eqs are to hold at a point (x,y), it follows that

$$2x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$$

and

$$2y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0$$

hence, a *necessary condition* for f to be differentiable at z is

$$z = x + jy = 0$$

(if $z \neq 0$ then f is not differentiable at z)

Cauchy-Riemann equations in Polar form

let $z = x + jy = re^{j\theta} \neq 0$ with $x = r\cos\theta$ and $y = r\sin\theta$

apply the Chain rule

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}\cdot r\sin\theta + \frac{\partial u}{\partial y}\cdot r\cos\theta$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\cdot r\sin\theta + \frac{\partial v}{\partial y}\cdot r\cos\theta$$

substitute $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (Cauchy-Riemanns equations)

the Cauchy-Riemann equations in the polar form are

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

example: Cauchy-Riemann eqs are satisfied but f' does not exist at $z = 0$

$$f(z) = \begin{cases} \bar{z}^2/z, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$$

from a direct calculation, express f as $f = u(x, y) + jv(x, y)$ where

$$u(x,y) = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2 + y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}, \qquad v(x,y) = \begin{cases} \dfrac{y^3 - 3x^2y}{x^2 + y^2}, & (x,y) \neq 0 \\ 0, & (x,y) = 0 \end{cases}$$

and we can say that

$$u(x,0) = x, \ \forall x, \quad u(0,y) = 0, \ \forall y, \quad v(x,0) = 0, \ \forall x, \quad v(0,y) = y, \ \forall y$$

which give

$$\frac{\partial u(x,0)}{\partial x} = 1, \ \forall x, \quad \frac{\partial u(0,y)}{\partial y} = 0, \ \forall y, \quad \frac{\partial v(x,0)}{\partial x} = 0, \ \forall x, \quad \frac{\partial v(0,y)}{\partial y} = 1, \ \forall y$$

so the Cauchy-Riemann equations are satisfied at $(x, y) = (0, 0)$

however, f is **not** differentiable at 0 because

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\frac{(\overline{\Delta z})^2}{\Delta z} - 0}{\Delta z} = \lim_{\Delta z \to 0} \frac{(\overline{\Delta z})^2}{(\Delta z)^2}$$

and the limit does not exist (from page 9-11)

Sufficient conditions for differentiability

✌ **Theorem:** let $z = x + jy$ and let the function

$$f(z) = u(x,y) + jv(x,y)$$

be defined on some neighborhood of z, and suppose that

1. the first partial derivatives of u and v w.r.t. x and y exist
2. the partial derivatives are **continuous** at (x,y) and satisfy **C-R eqs**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x,y)$$

then $f'(z)$ exists and its value is

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + j\frac{\partial v}{\partial x}(x,y)$$

example 1: on page 9-27, $f'(0)$ does not exist while the C-R eqs hold because

$$\frac{\partial u(x,y)}{\partial x} = \frac{x^4 - 3y^4 + 6x^2y^2}{(x^2+y^2)^2} \quad \Longrightarrow \quad \frac{\partial u(x,0)}{\partial x} = 1, \quad \frac{\partial u(0,y)}{\partial x} = -3$$

which show that $\frac{\partial u}{\partial x}$ is *not continuous* at $(x,y) = (0,0)$ (neither is $\frac{\partial v}{\partial y}$)

example 2: $f(z) = z^2 = x^2 - y^2 + j2xy$, find $f'(z)$ if it exists

check the Cauchy-Riemann eqs,

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

and all the partial derivatives are continuous at (x,y)

thus, $f'(z)$ exists and

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = 2x + j2y = 2z$$

example 3: $f(z) = e^z$, find $f'(z)$ if it exists

write $f(z) = e^x \cos y + je^x \sin y$

check the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$$

and all the derivatives are continuous for all (x, y)

thus $f'(z)$ exists everywhere and

$$f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = e^x \cos y + je^x \sin y$$

note that $f'(z) = e^z = f(z)$ for all z

Analytic functions

Definition: f is said to be **analytic** at z_0 if it has a derivative at z_0 and every point in some neighborhood of z_0

- the terms **regular** and **holomorphic** are also used to denote analyticity
- we say f is analytic on a domain D if it has a derivative *everywhere* in D
- if f is analytic at z_0 then z_0 is called a **regular** point of f
- if f is not analytic at z_0 but is analytic at some point in every neighborhood of z_0 then z_0 is called a **singular** point of f
- a function that is analytic at *every point* in the complex plane is called **entire**

let $f(z) = u(x,y) + jv(x,y)$ be defined on a domain D

✌ **Theorem:** $f(z)$ is analytic on D *if and only if* all of followings hold

- $u(x,y)$ and $v(x,y)$ have *continuous* first-order partial derivatives
- the Cauchy-Riemann equations are satisfied

examples ✎

- $f(z) = z$ is analytic everywhere (f is entire)
- $f(z) = \bar{z}$ is not analytic everywhere because

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

more examples ✎

- $f(z) = e^z = e^x \cos x + je^x \sin y$ is analytic everywhere (f is entire)

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

  and all the partial derivatives are continuous

- $f(z) = (z+1)(z^2+1)$ is analytic on **C** (f is entire)
- $f(z) = \dfrac{(z^3+1)}{(z^2-1)(z^2+4)}$ is analytic on **C** except at

$$z = \pm 1, \quad \text{and} \quad z = \pm j2$$

- $f(z) = xy + jy$ is not analytic everywhere because

$$\frac{\partial u}{\partial x} = y \neq 1 = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = x \neq 0 = -\frac{\partial v}{\partial x}$$

Theorem on analytic functions

let f be an analytic function everywhere in a domain D

Theorem: if $f'(z) = 0$ everywhere in D then $f(z)$ must be constant on D

Theorem: if $f(z)$ is real valued for all $z \in D$ then $f(z)$ must be constant on D

Harmonic functions

the equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

is called **Laplace's equation**

we say a function $u(x,y)$ is **harmonic** if

- the first- and second-order partial derivatives exist and are continuous
- $u(x,y)$ satisfy Laplace's equation

✎ **Theorem:** if $f(z) = u(x,y) + jv(x,y)$ is analytic in a domain D then u and v are harmonic in D

example: $f(z) = e^{-y}\sin x - je^{-y}\cos x$

- f is entire because

$$\frac{\partial u}{\partial x} = e^{-y}\cos x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -e^{-y}\sin x = -\frac{\partial v}{\partial x}$$

(C-R is satisfied for every (x, y) and the partial derivatives are continuous)

- we can verify that

$$\left.\begin{array}{rclcrcl} \dfrac{\partial u}{\partial x} & = & e^{-y}\cos x, & \qquad & \dfrac{\partial^2 u}{\partial x^2} & = & -e^{-y}\sin x \\ \dfrac{\partial u}{\partial y} & = & -e^{-y}\sin x, & \qquad & \dfrac{\partial^2 u}{\partial y^2} & = & e^{-y}\sin x \end{array}\right\} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- hence, $u(x, y) = e^{-y}\sin x$ is harmonic in every domain of the complex plane

Harmonic Conjugate

v is said to be a **harmonic conjugate** of u if

1. u and v are harmonic in a domain D
2. their first-order partial derivatives satisfy the Cauchy-Riemann equations on D

example: $f(z) = z^2 = x^2 - y^2 + j2xy$

- since f is entire, then u and v are harmonic on the complex plane
- since f is analytic, u and v satisfy the C-R equations
- therefore, v is a harmonic conjugate of u

✌ **Theorem:** $f(z) = u(x,y) + jv(x,y)$ is analytic in a domain D *if and only if*

v is a harmonic conjugate of u

example: $f = 2xy + j(x^2 - y^2)$

- f is not analytic anywhere because

$$\frac{\partial u}{\partial x} = 2y \neq -2y = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = 2x \neq -2x = -\frac{\partial v}{\partial x}$$

  (C-R eqs do not hold anywhere except $z = 0$)

- hence, $x^2 - y^2$ cannot be a harmonic conjugate of $2xy$ on any domain

(contrary to the example on page 9-38)

References

Chapter 2 in

J. W. Brown and R. V. Churchill, *Complex Variables and Applications*, 8th edition, McGraw-Hill, 2009

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T. W. Gamelin, *Complex Analysis*, Springer, 2001